SHARP BOUNDS OF THE THIRD HANKEL DETERMINANT FOR CLASSES OF UNIVALENT FUNCTIONS WITH BOUNDED TURNING

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Abstract. We improve the bounds of the third order Hankel determinant for two classes of univalent functions with bounded turning.

Keywords: analytic function; univalent function; Hankel determinant; upper bound; bounded turning

MSC 2020: 30C45, 30C50

1. Introduction and preliminaries

Univalent functions, which are functions which are analytic, one-on-one and onto a certain domain, play a significant role in geometric function theory and in complex analysis in general. Although the main problem in the area, the Bieberbach conjecture, was closed by de Branges in 1984, the theory of univalent functions still remains attractive. A concept from this theory that was recently rediscovered and finds its application in the theory of singularities (see [4]) and in the study of power series with integral coefficients, is the Hankel determinant of functions $f(z)=z+a_2z^2+a_3z^3+\ldots$ analytic in the unit disk $\mathbb{D}:=\{z\in\mathbb{C}\colon |z|<1\}$ for $q\geqslant 1$ and $n\geqslant 1$ defined by

$$H_q(n)=\begin{vmatrix} a_n & a_{n+1} & \ldots & a_{n+q-1}\\ a_{n+1} & a_{n+2} & \ldots & a_{n+q}\\ \vdots & \vdots & & \vdots\\ a_{n+q-1} & a_{n+q} & \ldots & a_{n+2q-2}\end{vmatrix}.$$

The class of all such functions is denoted by $\mathcal{A}$.

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The upper bound (preferebly sharp) of the modulus of the Hankel determinants has been extensively studied in recent time, mainly the second order case $H_2(2) = a_2a_4 - a_3^2$ and the third order case

$$H_3(1) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2).$$

This problem, as most others over the class of univalent functions, is difficult to tackle with for the general class, and instead its subclasses are studied. The best known result for the whole class $\mathcal{A}$ is the one of Hayman (see [6]) who showed that $|H_2(n)| \leqslant An^{1/2}$, where A is an absolute constant, and that this rate of growth is the best possible. For the subclasses, we list the results for the classes of starlike and convex functions

$$\mathcal{S}^* = \Big\{f \in \mathcal{A}\colon \operatorname{Re} \frac{zf'(z)}{f(z)} > 0,\, z \in \mathbb{D}\Big\}$$

and

$$\mathcal{C} = \Big\{f \in \mathcal{A}\colon \operatorname{Re}\Big(1 + \frac{zf''(z)}{f'(z)}\Big) > 0,\, z \in \mathbb{D}\Big\},$$

with the upper bound of the second Hankel determinant 1 and $\frac{1}{8}$ (see [8]), and of the third Hankel determinant $0.777987\ldots$ (see [13]) and $\frac{4}{135} = 0.0296\ldots$ (see [10]), respectively. The estimates for the second order determinant are sharp, while of the third order are not, but are best known. Other related results can be found in [2], [10], [12], [14], [15].

We will study the class $\mathcal{R} \subset \mathcal{A}$ of univalent functions satisfying

$$\operatorname{Re} f'(z) > 0, \quad z \in \mathbb{D}, \tag{1}$$

and the class $\mathcal{R}_1 \subset \mathcal{A}$ satisfying

$$\operatorname{Re}(f'(z) + zf''(z)) > 0, \quad z \in \mathbb{D}.$$

The functions from the class $\mathcal{R}$ are said to be of bounded turning since $\operatorname{Re} f'(z) > 0$ is equivalent to $|\arg f'(z)| < \frac{1}{2}\pi$, and $\arg f(z)$ is the angle of rotation of the image of a line segment starting from z under the mapping f. They are of special interest since they are not part of class of starlike functions which is very wide subclass of univalent functions. This is due to the counterexample by Krzyż (see [11]) showing that $\mathcal{S}^*$ does not contain $\mathcal{R}$, and $\mathcal{R}$ does not contain $\mathcal{S}^*$. In addition, classes $\mathcal{R}$ and $\mathcal{R}_1$ are related in the same way as the classes of starlike and convex functions, i.e. $\mathcal{R}_1 \subset \mathcal{R}$ (see [1]) as $\mathcal{C} \subset \mathcal{S}^*$, and

$$f \in \mathcal{R}_1 \Leftrightarrow zf'(z) \in \mathcal{R} \quad \text{as } f \in \mathcal{C} \Leftrightarrow zf'(z) \in \mathcal{S}^*.$$

For the class $\mathcal{R}$ in [7] the authors showed that

$$|H_2(2)| \leqslant \frac{4}{9} = 0.444\ldots,$$

and in [9] (with $\alpha = 1$ in Corollary 2.8),

$$|H_3(1)| \leqslant \frac{1}{540}\Big(\frac{877}{3} + 25\sqrt{5}\Big) = 0.64488\ldots$$

While the first estimate is sharp, the second one is not and we improve it here. We also give an upper bound of $H_3(1)$ for the class R_1.

For the study we use a different approach than the common one. In the current research on the upper bound of the Hankel determinant dominates a method based on a result on coefficients of Carathéodory functions (functions with positive real part on the unit disk) involving Toeplitz determinants. This result is due to Carathéodory and Toeplitz (see [14], Theorem 3.1.4, page 26) and its proof can be found in Grenander and Szegő (see [5]).

In this paper we use different method, based on the estimates of the coefficients of Schwartz functions. Here, it is a part of that result needed for the proofs.

Lemma 1.1. *Let $\omega(z) = c_1z + c_2z^2 + \ldots$ be a Schwarz function. Then for any real numbers μ and ν such that $(\mu,\nu) \in D_1 \cup D_2$, where*

$$D_1 = \Big\{(\mu,\nu)\colon\ |\mu| \leqslant \frac{1}{2},\ -1 \leqslant \nu \leqslant 1\Big\}$$

and

$$D_2 = \Big\{(\mu,\nu)\colon\ \frac{1}{2} \leqslant |\mu| \leqslant 2,\ \frac{4}{27}(|\mu|+1)^3 - (|\mu|+1) \leqslant \nu \leqslant 1\Big\},$$

the following sharp estimate holds

$$|c_3 + \mu c_1c_2 + \nu c_1^3| \leqslant 1.$$

We will also use the following, almost forgotten result of Carlson (see [3]).

Lemma 1.2. *Let $\omega(z) = c_1z + c_2z^2 + \ldots$ be a Schwarz function. Then*

$$|c_2| \leqslant 1 - |c_1|^2 \quad \text{and} \quad |c_4| \leqslant 1 - |c_1|^2 - |c_2|^2.$$

2. Main results

First we give the sharp estimate of the third Hankel determinant for the class $\mathcal{R}$.

Theorem 2.1. *Let $f \in \mathcal{R}$ be of the form $f(z) = z + a_2z^2 + a_3z^3 + \ldots$ Then*

$$|H_3(1)| \leqslant \frac{207}{540} = 0.38333\ldots$$

P r o o f. Condition (1) is equivalent to

$$f'(z) = \frac{1+\omega(z)}{1-\omega(z)},$$

i.e.

$$f'(z)(1-\omega(z)) = 1+\omega(z), \tag{2}$$

where ω is analytic in $\mathbb{D}$, $\omega(0) = 0$ and $|\omega(z)| < 1$ for all z in $\mathbb{D}$. If

$$\omega(z) = c_1z + c_2z^2 + \ldots,$$

then by equating the coefficients in (2), we have

$$\begin{aligned} a_2 &= c_1, \\ a_3 &= \frac{2}{3}(c_1^2 + c_2), \\ a_4 &= \frac{1}{2}(c_3 + 2c_1c_2 + c_1^3), \\ a_5 &= \frac{2}{5}(c_4 + 2c_1c_3 + 3c_1^2c_2 + c_1^4 + c_2^2). \end{aligned} \tag{3}$$

Using (3) we have

$$\begin{aligned} H_3(1) &= \frac{1}{540}(-12c_1^4c_2 - 16c_2^3 - 54c_1^3c_3 + 108c_1c_2c_3 - 135c_3^2 \\ &\quad + 60c_1^2c_2^2 - 7c_1^6 - 72c_1^2c_4 + 144c_2c_4) \\ &= \frac{1}{540}(-54c_3(c_3 - 2c_1c_2 + c_1^3) - 81c_3^2 - 12c_1^4c_2 - 16c_2^3 \\ &\quad + 60c_1^2c_2^2 - 7c_1^6 + 72(2c_2 - c_1^2)c_4) \end{aligned}$$

and

$$\begin{aligned} |H_3(1)| \leqslant \frac{1}{540}(&54|c_3||c_3 - 2c_1c_2 + c_1^3| + 81|c_3|^2 + 12|c_1|^4|c_2| + 16|c_2|^3 \\ &+ 60|c_1|^2|c_2|^2 + 7|c_1|^6 + 72(2|c_2| + |c_1|^2)|c_4|). \end{aligned} \tag{4}$$

If we choose $\mu = -2$ and $\nu = 1$ in Lemma 1.1, since $(\mu, \nu) \in D_2$, we receive that $|c_3 - 2c_1c_2 + c_1^3| \leqslant 1$. So, from (4) we get

$$(5) \qquad |H_3(1)| \leqslant \frac{1}{540}(54|c_3| + 81|c_3|^2 + 12|c_1|^4|c_2| + 16|c_2|^3 + 60|c_1|^2|c_2|^2 + 7|c_1|^6 + 72(2|c_2| + |c_1|^2)|c_4|).$$

Assume that $|c_2| \leqslant \frac{1}{2}(1 - |c_1|^2)$. Hence, $2|c_2| + |c_1|^2 \leqslant 1$. From this inequality and Lemma 1.2,

$$|H_3(1)| \leqslant \frac{1}{540}(54|c_3| + 81|c_3|^2 + 12|c_1|^4|c_2| + 16|c_2|^3 + 60|c_1|^2|c_2|^2 + 7|c_1|^6 + 72(1 - |c_1|^2 - |c_2|^2)),$$

so

$$\begin{aligned} |H_3(1)| &\leqslant \frac{1}{540}(72 + 54|c_3| + 81|c_3|^2 + 16|c_2|^2(|c_2| - 1) + 56|c_2|^2(|c_1|^2 - 1) \\ &\quad + 4|c_1|^2(|c_2|^2 - 1) + 7|c_1|^2(|c_1|^4 - 1) + 12|c_1|^2(|c_1|^2|c_2| - 1) - 49|c_1|^2) \\ &\leqslant \frac{1}{540}(72 + 54|c_3| + 81|c_3|^2), \end{aligned}$$

since all other terms are less or equal to zero because of $|c_1| \leqslant 1$ and $|c_2| \leqslant 1$ (see Lemma 1.2).

Providing that $|c_2| \leqslant \frac{1}{2}(1 - |c_1|^2)$ we have

$$|H_3(1)| \leqslant \frac{207}{540} = 0.38333\ldots$$

Now, assume that $\frac{1}{2}(1 - |c_1|^2) < |c_2| \leqslant (1 - |c_1|^2)$. Applying Lemma 1.2 in (5),

$$|H_3(1)| \leqslant \frac{1}{540}(54|c_3| + 81|c_3|^2 + 12|c_1|^4|c_2| + 16|c_2|^3 + 60|c_1|^2|c_2|^2 + 7|c_1|^6 + 72(2|c_2| + |c_1|^2)(1 - |c_1|^2 - |c_2|^2)).$$

From our assumption it follows that $2|c_2| + |c_1|^2 > 1$, so

$$|H_3(1)| \leqslant \frac{1}{540}(54|c_3| + 81|c_3|^2 + h_1(|c_1|^2, |c_2|)),$$

where

$$h_1(x, y) = 7x^3 - 72x^2 + 72x + 12x^2y - 12xy^2 - 144xy - 128y^3 + 144y,$$

$(x, y) \in D$ and D is such that $x + 2y > 1$, $x + y \leqslant 1$ and $x \geqslant 0$. But $-12xy^2 \leqslant 0$ and $7x^3 \leqslant 7$, so

$$h_1(x, y) < g_1(x, y) = -128y^3 + (144 - 144x + 12x^2)y - 72x^2 + 72x + 7.$$

It is enough to derive the greatest value of g_1 (even in the square $[0, 1] \times [0, 1]$). The critical points of g_1 satisfy the system of equations

$$\begin{cases} (x - 6)y + 3 - 6x = 0, \\ -32y^2 + (12 - 12x + x^2) = 0. \end{cases}$$

The first equation is contradictory if $x \in (\frac{1}{2}, 1]$. Suppose that $x \in [0, \frac{1}{2}]$. From this equation, $y = (6x - 3)/(x - 6)$. Putting it into the second one we obtain

$$12 - 12x + x^2 - 32\Big(\frac{6x - 3}{x - 6}\Big)^2 = 0,$$

or equivalently

$$144 + 480x(1 - 2x) + 6x(1 - 4x^2) + 90x + x^4 = 0,$$

which has no solutions in $[0, \frac{1}{2}]$.

On the boundary of the square $[0, 1] \times [0, 1]$ there is

$$\begin{aligned} g_1(x, 0) &= 7 + 72x - 72x^2 \leqslant 25, \\ g_1(x, 1) &= 23 - 72x - 60x^2 \leqslant 23, \\ g_1(1, y) &= 7 + 12y - 128y^3 \leqslant 7 + \sqrt{2}, \\ g_1(0, y) &= 7 + 144y - 128y^3 \leqslant 7 + 24\sqrt{6} = 65.787\ldots \end{aligned}$$

This means that in this case

$$H_3(1) \leqslant \frac{1}{540}(135 + 65.787\ldots) < \frac{207}{540}.$$

Summing up, $|H_3(1)| \leqslant \frac{207}{540}$. □

Now we give the estimate of the third Hankel determinant for the class $\mathcal{R}_1$.

Theorem 2.2. *Let $f \in \mathcal{R}_1$ be of the form $f(z) = z + a_2z^2 + a_3z^3 + \ldots$ Then*

$$|H_3(1)| \leqslant \frac{3537}{129600} = 0.02729\ldots$$

P r o o f. Similarly as in the proof of the previous theorem, for each function f from $\mathcal{R}_1$ there exists a function $\omega(z) = c_1 z + c_2 z^2 + \ldots$ analytic in $\mathbb{D}$ such that $|\omega(z)| < 1$ for all z in $\mathbb{D}$ and

$$f'(z) + zf''(z) = \frac{1 + \omega(z)}{1 - \omega(z)},$$

i.e.

$$(f'(z) + zf''(z))(1 - \omega(z)) = 1 + \omega(z).$$

Equating the coefficients in the previous expression leads to

$$\begin{aligned}
a_2 &= \frac{c_1}{2},\\
a_3 &= \frac{2}{9}(c_1^2 + c_2),\\
a_4 &= \frac{1}{8}(c_3 + 2c_1c_2 + c_1^3),\\
a_5 &= \frac{2}{25}(c_4 + 2c_1c_3 + 3c_1^2c_2 + c_1^4 + c_2^2).
\end{aligned}$$

From here, after some calculations we receive

$$\begin{aligned}
H_3(1) = {} & \frac{1}{1166400}(-1217c_1^6 - 1140c_1^4c_2 + 13116c_1^2c_2^2 + 7936c_2^3 - 9234c_1^3c_3\\
& + 972c_1c_2c_3 - 18225c_3^2 + 2592(8c_2 - c_1^2)c_4)\\
= {} & \frac{1}{1166400}\Big(-8991c_3^2 - 9234c_3\Big(c_3 - \frac{2}{19}c_1c_2 + c_1^3\Big) - 1140c_1^4c_2\\
& + 13116c_1^2c_2^2 + 7936c_2^3 - 1217c_1^6 + 2592(8c_2 - c_1^2)c_4\Big),
\end{aligned}$$

and further

$$\begin{aligned}
|H_3(1)| \leqslant {} & \frac{1}{1166400}\Big(8991|c_3|^2 + 9234|c_3|\Big|c_3 - \frac{2}{19}c_1c_2 + c_1^3\Big| + 1140|c_1|^4|c_2|\\
& + 13116|c_1|^2|c_2|^2 + 7936|c_2|^3 + 1217|c_1|^6 + 2592(8|c_2| + |c_1|^2)|c_4|\Big).
\end{aligned}$$

Now, for $\mu = -\frac{2}{19}$ and $\nu = 1$ in Lemma 1.1, we have $(\mu, \nu) \in D_1$ and $\big|c_3 - \frac{2}{19}c_1c_2 + c_1^3\big| \leqslant 1$, which implies

$$\begin{aligned}
|H_3(1)| \leqslant {} & \frac{1}{1166400}(9234|c_3| + 8991|c_3|^2 + 1140|c_1|^4|c_2| + 7936|c_2|^3\\
& + 13116|c_1|^2|c_2|^2 + 1217|c_1|^6 + 2592(8|c_2| + |c_1|^2)|c_4|).
\end{aligned}$$

Assume that $|c_2| \leqslant \frac{21}{32}(1-|c_1|^2)$. Hence, $8|c_2|+|c_1|^2 \leqslant \frac{1}{4}(21-17|c_1|^2)$. From this inequality and Lemma 1.2,

$$\begin{aligned}|H_3(1)| \leqslant \frac{1}{1166400}&(9234|c_3|+8991|c_3|^2+1140|c_1|^4|c_2|+7936|c_2|^3\\&+13116|c_1|^2|c_2|^2+1217|c_1|^6+648(21-17|c_1|^2)(1-|c_1|^2-|c_2|^2))\end{aligned}$$

and

$$\begin{aligned}|H_3(1)| &\leqslant \frac{1}{1166400}(13608+9234|c_3|+8991|c_3|^2+7936|c_2|^2(|c_2|-1)\\&\quad+7444|c_1|^2(|c_2|^2-1)+1140|c_1|^2(|c_1|^2|c_2|-1)+1217|c_1|^2(|c_1|^4-1)\\&\quad+5672|c_2|^2(|c_1|^2-1)-3807|c_1|^2-11016|c_1|^2(1-|c_1|^2-|c_2|^2))\\&\leqslant \frac{1}{1166400}(13608+9234|c_3|+8991|c_3|^2),\end{aligned}$$

since all other terms are less or equal to zero (again because of $|c_1| \leqslant 1$ and $|c_2| \leqslant 1$ which follows from Lemma 1.2).

The greatest value of the function in brackets is attained for $|c_3|=1$ and it is equal to 31833. In this way we have proven that

$$|H_3(1)| \leqslant \frac{31833}{11664000} = \frac{3537}{129600} = 0.02729\ldots$$

under the condition $|c_2| \leqslant \frac{21}{32}(1-|c_1|^2)$.

Assume now that $\frac{21}{32}(1-|c_1|^2) < |c_2| \leqslant (1-|c_1|^2)$. From Lemma 1.2,

$$\begin{aligned}|H_3(1)| \leqslant \frac{1}{1166400}&(9234|c_3|+8991|c_3|^2+1140|c_1|^4|c_2|+7936|c_2|^3\\&+13116|c_1|^2|c_2|^2+1217|c_1|^6+2592(8|c_2|+|c_1|^2)(1-|c_1|^2-|c_2|^2)).\end{aligned}$$

From the assumption it follows that $8|c_2|+|c_1|^2 > \frac{1}{4}(21-17|c_1|^2)$ and

$$\begin{aligned}|H_3(1)| &\leqslant \frac{1}{1166400}(9234|c_3|+8991|c_3|^2+h_2(|c_1|^2,|c_2|))\\&\leqslant \frac{1}{1166400}(18225+h_2(|c_1|^2,|c_2|)),\end{aligned}$$

where

$$\begin{aligned}h_2(x,y) = &-12800y^3+10524xy^2+(1140x^2-20736x+20736)y\\&+1217x^3-2592x^2+13608x\end{aligned}$$

and $(x,y) \in D$, D is such that $x+\frac{21}{32}y > 1$, $x+y \leqslant 1$ and $x \geqslant 0$.

We shall derive the greatest value of h_2 in $E = \{(x,y)\colon\ x \geqslant 0,\ y \geqslant 0,\ x+y \leqslant 1\}$, i.e. in the superset of D. Note that

$$\begin{aligned}\frac{\partial h_2}{\partial x} &= 3(3508y^2 - 6912y + 760xy + 1217x^2 - 1728x + 4536)\\ &= 3(760(1-x)(1-y) + 3076(1-y)^2 + 484(1-x)^2 + 216 + 733x^2 + 432y^2) \geqslant 0.\end{aligned}$$

It means that the greatest value of h_2 is obtained on the boundary of E. We have

$$\begin{aligned}h_2(x,0) &= 1217x^3 + 11016x + 2592x(1-x) \leqslant 1217x^3 + 11016x \leqslant 12233,\\ h_2(0,y) &= 20736y - 12800y^2 \leqslant \frac{209952}{25} = 8398.08\ldots\end{aligned}$$

Additionally, it is not difficult to show that

$$h_2(x, 1-x) = 7936 + 21060x - 40164x^2 + 23401x^3 \leqslant 12233.$$

Hence, in this case,

$$H_3(1) \leqslant \frac{1}{1166400}(18225 + 12233) = \frac{15229}{583200} = 0.02611\ldots$$

Summing up, $|H_3(1)| \leqslant \frac{3537}{129600} = 0.02729\ldots$ $\square$

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