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ON THIRD ORDER HANKEL DETERMINANT FOR INVERSE FUNCTIONS OF CERTAIN CLASSES OF UNIVALENT FUNCTIONS

MILUTIN OBRADOVIĆ[1], NIKOLA TUNESKI[2,*]

Abstract. In this paper we determine the upper bounds for the Hankel determinant of third order for the inverse functions of functions from some classes of univalent functions.

1. Introduction and preliminaries

Let $\mathcal{A}$ be the class containing functions that are analytic in the unit disk $\mathbb{D} := |z| < 1$ and are normalized such that $f(0) = 0 = f'(0) - 1$, i.e.,

$$f(z) = z + a_2z^2 + a_3z^3 + \cdots . \tag{1}$$

By $\mathcal{S}$ we denote the class of functions from $\mathcal{A}$ which are univalent in $\mathbb{D}$.

A problem that recently rediscovered, is to find upper bound (preferably sharp) of the modulus of the Hankel determinant $H_q(n)(f)$ of a given function f, for $q \geq 1$ and $n \geq 1$, defined by

$$H_q(n)(f) = \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \dots & a_{n+q} \\ \vdots & \vdots & & \vdots \\ a_{n+q-1} & a_{n+q} & \dots & a_{n+2q-2} \end{vmatrix}.$$

The general Hankel determinant is hard to deal with, so the second and the third ones,

$$H_2(2)(f) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2$$

and

$$H_3(1)(f) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2),$$

respectively, are studied instead. The research is focused on the subclasses of univalent functions (starlike, convex, α-convex, close-to-convex, spirallike,...) since the general class of normalised

[1]Department of Mathematics, Faculty of Civil Engineering, University of Belgrade, Bulevar Kralja Aleksandra 73, 11000, Belgrade, Serbia

[2]Department of Mathematics and Informatics, Faculty of Mechanical Engineering, Ss. Cyril and Methodius University in Skopje, Karpoš II b.b., 1000 Skopje, Republic of North Macedonia

[*]Corresponding author

E-mail address: obrad@grf.bg.ac.rs, nikola.tuneski@mf.ukim.edu.mk.

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univalent functions is also hard to deal with. Some of the more significant results can be found in [2–8, 10].

In this paper we will give the upper bound of the modulus of the third Hankel determinant for the inverse functions for the functions in different subclasses of $\mathcal{S}$ as listed bellow.

The classes $\mathcal{R}, \mathcal{C}, \mathcal{S}^\star, \mathcal{S}^\star_s$ (with bounded turning, convex, starlike and starlike with respect to symmetric points, respectively) are defined in the following way:

$$\begin{aligned}
\mathcal{R} &= \left[f \in \mathcal{A} : \mathrm{Re} f'(z) > 0,\ z \in \mathbb{D}\right],\\
\mathcal{C} &= \left[f \in \mathcal{A} : \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0,\ z \in \mathbb{D}\right],\\
\mathcal{S}^\star &= \left[f \in \mathcal{A} : \mathrm{Re}\frac{zf'(z)}{f(z)} > 0,\ z \in \mathbb{D}\right],\\
\mathcal{S}^\star_s &= \left[f \in \mathcal{A} : \mathrm{Re}\frac{2zf'(z)}{f(z) - f(-z)} > 0,\ z \in \mathbb{D}\right].
\end{aligned}$$

For every univalent function in $\mathbb{D}$, exists inverse at least on the disk with radius $1/4$ (due to the famous Koebe's $1/4$ theorem). If the inverse has an expansion

$$f^{-1}(w) = w + A_2 w^2 + A_3 w^3 + \cdots, \tag{2}$$

then, by using the identity $f(f^{-1}(w)) = w$, from (1) and (2) we receive

$$\begin{aligned}
A_2 &= -a_2,\\
A_3 &= -a_3 + 2a_2^2,\\
A_4 &= -a_4 + 5a_2a_3 - 5a_2^3,\\
A_5 &= -a_5 + 6a_2a_4 - 21a_2^2a_3 + 3a_3^2 + 14a_2^4.
\end{aligned} \tag{3}$$

As for the Hankel determinant of the third order for inverse functions, by using the definition of $H_3(1)(f)$ and the relations (3), after some calculations, we receive:

$$\begin{aligned}
H_3(1)(f^{-1}) &= A_3(A_2A_4 - A_3^2) - A_4(A_4 - A_2A_3) + A_5(A_3 - A_2^2)\\
&= a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2) - (a_3 - a_2^2)^3\\
&= H_3(1)(f) - (a_3 - a_2^2)^3,
\end{aligned}$$

i.e.,

$$H_3(1)(f^{-1}) = H_3(1)(f) - (a_3 - a_2^2)^3, \tag{4}$$

and also,

$$|H_3(1)(f^{-1}) - H_3(1)(f)| = |a_3 - a_2^2|^3. \tag{5}$$

For our consideration we need the next lemmas.

Lemma 1 ([1]). *Let*

$$\omega(z) = c_1 z + c_2 z^2 + \cdots \tag{6}$$

be a Schwartz function, i.e., a function analytic in $\mathbb{D}$, $\omega(0) = 0$ *and* $|\omega(z)| < 1$. *Then*

$$|c_1| \leq 1, \quad |c_2| \leq 1 - |c_1|^2,$$
$$|c_3| \leq 1 - |c_1|^2 - \frac{|c_2|^2}{1+|c_1|}, \quad \textit{and} \quad |c_4| \leq 1 - |c_1|^2 - |c_2|^2.$$

Lemma 2 ([9]). *Let* ω *given by* (6) *be a Schwartz function. Then, for any real numbers* μ *and* ν *from the sets* D_1 *and* D_2, *the following sharp estimate holds*

$$|c_3 + \mu c_1 c_2 + \nu c_1^3| \leq 1,$$

where

$$D_1 = \left\{ (\mu, \nu) : |\mu| \leq \frac{1}{2}, \, -1 \leq \nu \leq 1 \right\},$$
$$D_2 = \left\{ (\mu, \nu) : \frac{1}{2} \leq |\mu| \leq 2, \, \frac{4}{27}(|\mu|+1)^3 - (|\mu|+1) \leq \nu \leq 1 \right\}.$$

2. Main results

Theorem 1. *Let* $f \in \mathcal{R}$. *Then we have*

$$|H_3(1)(f^{-1})| \leq 0.593155\ldots.$$

Proof. Since $f \in \mathcal{R}$ is equivalent to

$$f'(z) = \frac{1+\omega(z)}{1-\omega(z)},$$

where ω is a Schwartz function, then

$$f'(z) = 1 + 2\omega(z) + 2\omega^2(z) + \cdots. \tag{7}$$

Using the notations for f and ω given by (1) and (6) and equating the coefficients in (7), we have

$$\begin{cases} a_2 = c_1, \\ a_3 = \frac{2}{3}(c_2 + c_1^2) \\ a_4 = \frac{1}{2}(c_3 + 2c_1c_2 + c_1^3), \\ a_5 = \frac{2}{5}(c_4 + 2c_1c_3 + 3c_1^2c_2 + c_2^2 + c_1^4). \end{cases} \tag{8}$$

Now, from (4) and (8), and some computations, for $f \in \mathcal{R}$ we obtain:

$$\begin{aligned} H_3(1)(f^{-1}) = \frac{1}{540} \big[&-135c_3^2 + 108c_1c_2c_3 - 54c_1^3c_3 + 300c_1^2c_2^2 \\ &-132c_1^4c_2 - 176c_2^3 + 13c_1^6 + 72(2c_2 - c_1^2)c_4 \big]. \end{aligned} \tag{9}$$

From (9), after some rearrangements we have

$$\begin{aligned} &540 \cdot H_3(1)(f^{-1}) \\ &= -135c_3\left(c_3 + \frac{8}{45}c_1c_2 + \frac{2}{5}c_1^3\right) + 132c_1c_2\left(c_3 + \frac{1}{2}c_1c_2 - c_1^3\right) \\ &+ 234c_1^2c_2^2 - 176c_2^3 + 13c_1^6 + 72(2c_2 - c_1^2)c_4, \end{aligned}$$

and from here

$$\begin{aligned}
&540\left|H_3(1)(f^{-1})\right| \\
&\le 135|c_3|\left|c_3+\frac{8}{45}c_1c_2+\frac{2}{5}c_1^3\right|+132|c_1||c_2|\left|c_3+\frac{1}{2}c_1c_2-c_1^3\right| \\
&+234|c_1|^2|c_2|^2+176|c_2|^3+13|c_1|^6+72\left(2|c_2|+|c_1|^2\right)|c_4|.
\end{aligned}$$

Using the results of Lemma 2 (case D_1) and Lemma 1 for $|c_4|$, from the last relation we have

$$\begin{aligned}
&540\left|H_3(1)(f^{-1})\right| \\
&\le 135|c_3|+132|c_1||c_2|+234|c_1|^2|c_2|^2+176|c_2|^3+13|c_1|^6 \\
&\quad+72\left(2|c_2|+|c_1|^2\right)\left(1-|c_1|^2-|c_2|^2\right) \\
&=135|c_3|+132|c_1||c_2|+176|c_1|^2|c_2|^2+32|c_2|^3+13|c_1|^6 \\
&\quad+144|c_2|-144|c_1|^2|c_2|+72|c_1|^2-72|c_1|^4.
\end{aligned}$$

Using Lemma 2 we have next estimation for $|c_3|$:

$$\begin{aligned}
|c_3| &\le 1-|c_1|^2-\frac{|c_2|^2}{1+|c_1|} \\
&=1-|c_1|^2-|c_2|^2+\frac{|c_1||c_2|^2}{1+|c_1|} \\
&\le 1-|c_1|^2-|c_2|^2+\frac{|c_1|}{1+|c_1|}\left(1-|c_1|^2\right)^2,
\end{aligned}$$

i.e.,

$$|c_3| \le (1-|c_1|^2)^2+|c_1|\left(1-|c_1|^2\right)-|c_2|^2. \tag{10}$$

If we use the estimation given in (10) in the last relation for $540\left|H_3(1)(f^{-1})\right|$, then we have

$$\begin{aligned}
540\left|H_3(1)(f^{-1})\right| &\le 135(1-|c_1|^2)^2+135|c_1|(1-|c_1|^2)-135|c_2|^2 \\
&\quad+132|c_1||c_2|+176|c_1|^2|c_2|^2+32|c_2|^3+13|c_1|^6 \\
&\quad+144|c_2|-144|c_1|^2|c_2|+72|c_1|^2-72|c_1|^4 \\
&=135(1-|c_1|^2)^2+135|c_1|(1-|c_1|^2)+132|c_1||c_2| \\
&\quad+176|c_1|^2|c_2|^2+13|c_1|^6+144|c_2|-144|c_1|^2|c_2| \\
&\quad+72|c_1|^2-72|c_1|^4-|c_2|^2(135-32|c_2|)
\end{aligned}$$

Finally, using the estimation $|c_2| \le 1-|c_1|^2$ and some calculations, from the last relation we obtain that

$$\begin{aligned}
540\left|H_3(1)(f^{-1})\right| &\le 175|c_1|^6-117|c_1|^4-267|c_1|^3-324|c_1|^2+267|c_1|+279 \\
&\le 320.30\ldots,
\end{aligned}$$

since the function on the right hand side of $|c_1|$ has maximum value $320.30\ldots$ in the interval $[0,1]$, obtained for $|c_1|=0.23887\ldots$. So, for $f\in\mathcal{R}$:

$$\left|H_3(1)(f^{-1})\right| \le \frac{320.30\ldots}{540}=0.59315\ldots.$$

□

Remark 1. *Since for $f \in \mathcal{R}$, by using the relation (8), we have*

$$\left|a_3 - a_2^2\right|^3 = \left|\frac{2}{3}(c_2 + c_1^2) - c_1^2\right|^3 \leq \frac{1}{27}\left(2|c_2| + |c_1|^2\right)^3 \leq \frac{1}{27}\left(2 - |c_1|^2\right)^3 \leq \frac{8}{27},$$

where the equality attains attains for $|c_1| = 0$ and $|c_2| = 1$. In that sense, from (5) we obtain

$$\left|H_3(1)(f^{-1}) - H_3(1)(f)\right| \leq \frac{8}{27},$$

and the result is sharp. The extremal function is defined by $f'(z) = \frac{1+z^2}{1-z^2}$, i.e., by $f(z) = -z + \ln\frac{1+z}{1-z}$.

Theorem 2. *Let $f \in \mathcal{C}$, then*

$$\left|H_3(1)(f^{-1})\right| \leq \frac{11}{180} = 0.061\ldots.$$

Proof. From the definition of the class $\mathcal{C}$ we have

$$1 + \frac{zf''(z)}{f'(z)} = \frac{1+\omega(z)}{1-\omega(z)}, \tag{11}$$

where ω is a Schwartz function, and from here

$$(zf'(z))' = \left[1 + 2\left(\omega(z) + \omega^2(z) + \cdots\right)\right] \cdot f'(z). \tag{12}$$

Using the notations (1) and (6), and comparing the coefficients in the relation (12), then, after some simple calculations, we obtain

$$\begin{cases} a_2 = c_1, \\ a_3 = \frac{1}{3}\left(c_2 + 3c_1^2\right), \\ a_4 = \frac{1}{6}\left(c_3 + 5c_1c_2 + 6c_1^3\right) \\ a_5 = \frac{1}{30}\left(3c_4 + 14c_1c_3 + 43c_1^2c_2 + 30c_1^4 + 6c_2^2\right). \end{cases} \tag{13}$$

From the relations (4) and (13), and some transformations, we get

$$H_3(1)(f^{-1}) = \frac{1}{540}\left(-6c_1c_2c_3 + 3c_1^2c_2^2 - 4c_2^3 - 15c_3^2 + 18c_2c_4\right).$$

From the previous relation we have

$$540 \cdot H_3(1)(f^{-1}) = -15\left(c_3 + \frac{1}{5}c_1c_2\right)^2 + \frac{18}{5}c_1^2c_2^2 - 4c_2^3 + 18c_2c_4,$$

and from here

$$\begin{aligned} 540\left|H_3(1)(f^{-1})\right| &\leq 15\left|c_3 + \frac{1}{5}c_1c_2\right|^2 + \frac{18}{5}|c_1|^2|c_2|^2 + 4|c_2|^3 + 18|c_2||c_4| \\ &\leq 15 + \frac{18}{5}|c_1|^2|c_2|^2 + 4|c_2|^3 + 18|c_2|(1 - |c_1|^2 - |c_2|^2) \\ &= 15 + 18|c_2| - \frac{18}{5}|c_1|^2|c_2|(5 - |c_2|) - 14|c_2|^3 \\ &\leq 15 + 18|c_2| \\ &\leq 33, \end{aligned}$$

where we used Lemma 1 and Lemma 2. From the previous relations we have the statement of the theorem. □

Remark 2. *In the case when $f \in \mathcal{C}$, by using the relation (13), we have*

$$\left|a_3 - a_2^2\right|^3 = \frac{1}{27}|c_2|^3 \le \frac{1}{27},$$

with equality for $|c_2| = 1$. This, using (5), implies that

$$\left|H_3(1)(f^{-1}) - H_3(1)(f)\right| \le \frac{1}{27},$$

with sharpness for the function defined by (11) and $\omega(z) = z^2$.

Theorem 3. *Let $f \in \mathcal{S}^\star$. Then*

$$\left|H_3(1)(f^{-1}) - H_3(1)(f)\right| \le 1,$$

with equality sign for the Koebe function, and

$$|H_3(1)(f^{-1})| < \frac{14}{9} = 1.555\ldots.$$

Proof. For $f \in \mathcal{S}^\star$ we have $|a_3 - a_2^2| \le 1$ (as for the class $\mathcal{S}$), so that from (5):

$$\left|H_3(1)(f^{-1}) - H_3(1)(f)\right| \le 1.$$

Further, using that for $f \in \mathcal{S}^\star$, $|H_3(1)(f)| < \frac{5}{9}$ ([11, Theorem 1]), we have

$$\left|H_3(1)(f^{-1})\right| \le 1 + |H_3(1)(f)| < 1 + \frac{5}{9} = \frac{14}{5}.$$

□

Theorem 4. *Let $f \in \mathcal{S}_s^\star$. Then*

$$\left|H_3(1)(f^{-1})\right| \le \frac{5}{4}.$$

Proof. Using the definition of the class $\mathcal{S}_s^\star$ we have that there exists a Schwartz function ω such that

$$\frac{2zf'(z)}{f(z) - f(-z)} = \frac{1 + \omega(z)}{1 - \omega(z)}, \tag{14}$$

and from here

$$2zf'(z) = \left[1 + 2\left(\omega(z) + \omega^2(z) + \cdots\right)\right] \cdot [f(z) - f(-z)]. \tag{15}$$

Similarly, as in two previous cases (in Theorem 1 and Theorem 2), by comparing the coefficients in the relation (15), after some simple calculations, we have

$$\begin{cases} a_2 = c_1 \\ a_3 = c_2 + c_1^2 \\ a_4 = \frac{1}{2}\left(c_3 + 3c_1c_2 + 2c_1^3\right) \\ a_5 = \frac{1}{2}\left(c_4 + 2c_1c_3 + 5c_1^2c_2 + 2c_1^4 + 2c_2^2\right). \end{cases} \tag{16}$$

Now,from (4) and (16) and some calculations, for $f \in \mathcal{S}_s^\star$ we obtain:

$$\begin{aligned} H_3(1)(f^{-1}) &= \frac{1}{4}\left(-c_3^2 + 2c_1c_2c_3 + c_1^2c_2^2 - 4c_2^3 + 2c_2c_4\right) \\ &= \frac{1}{4}\left[-(c_3 - c_1c_2)^2 + 2c_1^2c_2^2 - 4c_2^3 + 2c_2c_4\right], \end{aligned}$$

which implies

$$\left|H_3(1)(f^{-1})\right| \le \frac{1}{4}\left(|c_3 - c_1c_2|^2 + 2|c_1|^2|c_2|^2 + 4|c_2|^3 + 2|c_2||c_4|\right).$$

From the last relation and Lemma 2 (case D_2) and Lemma 1, we have:

$$\begin{aligned}\left|H_3(1)(f^{-1})\right| &\le \frac{1}{4}\left[1 + 2|c_1|^2|c_2|^2 + 4|c_2|^3 + 2|c_2|\left(1 - |c_1|^2 - |c_2|^2\right)\right] \\ &= \frac{1}{4}\left[1 + 2|c_2| + 2|c_2|^3 - 2|c_1|^2|c_2|(1 - |c_2|)\right] \\ &\le \frac{1}{4}\left(1 + 2|c_2| + 2|c_2|^3\right) \\ &\le \frac{5}{4}.\end{aligned}$$

□

Remark 3. *For $f \in \mathcal{S}_s^\star$ we have $|a_3 - a_2^2|^3 = |c_2|^3 \le 1$, so that from (5):*

$$\left|H_3(1)(f^{-1}) - H_3(1)(f)\right| \le 1.$$

The equality is attained for the function f defined by

$$\frac{2zf'(z)}{f(z) - f(-z)} = \frac{1+z^2}{1-z^2},$$

i.e., with $\omega(z) = z^2$ in (14).

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