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# ON A SPECIAL CLASS OF SCHWARTZ FUNCTIONS

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ABSTRACT. In this paper we study functions $\omega(z) = c_1z+c_2z^2+c_3z^3+\cdots$ analytic in the open unit disk $\mathbb{D}$ and such that $|\omega'(z)| \le 1$ for all $z \in \mathbb{D}$. For these functions we give estimates (sometimes sharp) for the following moduli: $|c_3 - c_1c_2|$, $|c_1c_3 - c_2^2|$, and $|c_4 - c_2^2|$.

## 1. INTRODUCTION AND DEFINITIONS

For a function $\omega$, analytic in the open unit disk $\mathbb{D} = \{z : |z| < 1\}$ and of the form

$$\omega(z) = c_1z + c_2z^2 + c_3z^3 + \cdots , \qquad (c_1, c_2, \ldots \in \mathcal{C}) \tag{1.1}$$

we say that is Schwartz function if $|\omega(z)| < 1$, $z \in \mathbb{D}$. We denote by $\mathcal{B}_0$ the class of all such functions.

In his paper [4], Zaprawa gave many different inequalities for the coefficients $c_1, c_2, \ldots$ for the functions of the class $\mathcal{B}_0$.

In this paper we study the class of functions $\mathcal{B}_0'$ of type (1.1) such that $|\omega'(z)| \le 1$ for all $z \in \mathbb{D}$. Since

$$z\omega'(z) = c_1z + 2c_2z^2 + 3c_3z^3 \cdots , \tag{1.2}$$

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and $|z\omega'(z)| = |z| \cdot |\omega'(z)| \leq |z| < 1$, $z \in \mathbb{D}$, it means that $z\omega'(z)$ belongs to $\mathcal{B}_0$. Also, since $\omega(z) = \int_0^z \omega'(z)dz$, then $|\omega(z)| \leq \int_0^{|z|} |\omega'(z)|\, d|z| \leq |z| < 1$ for all $z \in \mathbb{D}$, i.e., $\omega \in \mathcal{B}_0$. So, $|\omega'(z)| \leq 1$, $z \in \mathbb{D}$, is a sufficient condition for $\omega \in \mathcal{B}_0$, i.e., $\mathcal{B}_0'$ is subclass of the class $\mathcal{B}_0$.

For the functions from $\mathcal{B}_0'$ we try to find properties for the coefficients $c_1, c_2, c_3, \ldots$ that correspond to the properties for the functions from $\mathcal{B}_0$.

For our considerations we will need the next lemma originating from [1].

**Lemma 1.1.** *Let* $\omega \in \mathcal{B}_0$ *is given by* (1.1). *Then*

$$\begin{gathered} |c_1| \leq 1, \qquad |c_2| \leq 1 - |c_1|^2, \\ |c_3| \leq 1 - |c_1|^2 - \tfrac{|c_2|^2}{1+|c_1|}, \\ |c_4| \leq 1 - |c_1|^2 - |c_2|^2. \end{gathered} \tag{1.3}$$

We showed that when $\omega$ given by (1.1) is in $\mathcal{B}_0$, then $z\omega'(z)$ is in $\mathcal{B}_0'$. Thus, Lemma 1.1, together with (1.2), directly brings

**Lemma 1.2.** *Let* $\omega \in \mathcal{B}_0'$ *is given by* (1.1). *Then*

$$\begin{gathered} |c_1| \leq 1, \qquad |c_2| \leq \frac{1}{2}\left(1 - |c_1|^2\right), \\ |c_3| \leq \frac{1}{3}\left(1 - |c_1|^2 - \frac{4|c_2|^2}{1+|c_1|}\right), \\ |c_4| \leq \frac{1}{4}\left(1 - |c_1|^2 - 4|c_2|^2\right). \end{gathered} \tag{1.4}$$

## 2. Main results

We begin with partly sharp estimate of the modulus $|c_3 - c_1c_2|$ for functions from $\mathcal{B}_0'$ with expansion (1.1).

**Theorem 2.1.** *If* $\omega \in \mathcal{B}_0'$ *is of form* (1.1)*, then*

$$|c_3 - c_1c_2| \leq \begin{cases} \frac{1}{48}(1+|c_1|)\left[9|c_1|^2 - 16|c_1| + 16\right], & 0 \leq |c_1| \leq \frac{4}{7} \\ \frac{5}{6}|c_1|(1-|c_1|^2), & \frac{4}{7} \leq |c_1| \leq 1 \end{cases}. \tag{2.1}$$

*The estimate is sharp for* $|c_1| = 0$ *and for* $\frac{4}{7} \leq |c_1| \leq 1$.

*Proof.* For $\omega \in \mathcal{B}_0'$ and $\omega$ given by (1.1) we apply the inequalities (1.4):

$$|c_3 - c_1c_2| \le |c_3| + |c_1||c_2| \le \frac{1}{3}\left(1 - |c_1|^2 - \frac{4|c_2|^2}{1+|c_1|}\right) + |c_2||c_2|$$
$$= -\frac{4}{3(1+|c_1|)}|c_2|^2 + |c_1||c_2| + \frac{1}{3}(1-|c_1|^2).$$

If we consider the last expression as a function of $|c_2|$, $0 \le |c_2| \le \frac{1}{2}(1-|c_1|^2)$, then we easily obtain the estimate given by (2.1), depending on its maximum which in the case $0 \le |c_1| \le \frac{4}{7}$ is attained for $|c_2| = \frac{3}{8}|c_1|(1+|c_1|)$ lying in the interval $\left(0, \frac{1}{2}(1-|c_1|^2)\right)$, and in the case $\frac{4}{7} \le |c_1| \le 1$ is attained for $|c_2| = \frac{1}{2}(1-|c_1|^2)$.

For $|c_1| = 0$ and for $\frac{4}{7} \le |c_1| \le 1$ the result is sharp with extremal functions $\omega_1(z) = \frac{1}{3}z^3$ and

$$\omega_2(z) = \int_0^z \frac{|c_1| + z}{1+|c_1|z}\, dz = |c_1|z + \frac{1}{2}(1-|c_1|^2)z^2 - \frac{1}{3}|c_1|(1-|c_1|^2)z^3 + \cdots,$$

respectively. □

**Remark 2.1.** *Theorem 2.1 brings:*

$$\omega \in \mathcal{B}_0' \qquad \Rightarrow \qquad |c_3 - c_1c_2| \le \frac{1}{3},$$

*while*

$$\omega \in \mathcal{B}_0 \qquad \Rightarrow \qquad |c_3 - c_1c_2| \le 1$$

*follows from [4].*

Similarly as Theorem 2.1 we can prove the next theorem.

**Theorem 2.2.** *If $\omega \in \mathcal{B}_0'$ is of form (1.1) and $\mu \in \mathcal{C}$, then*

$$\text{(2.2)}\quad |c_3 - \mu c_1c_2| \le \begin{cases} \frac{1}{48}(1+|c_1|)\left[9|\mu|^2|c_1|^2 - 16|c_1| + 16\right], & 0 \le |c_1| \le \frac{1}{1+3/4|\mu|} \\ \left(\frac{1}{3} + \frac{1}{2}|\mu|\right)|c_1|(1-|c_1|^2), & \frac{1}{1+3/4|\mu|} \le |c_1| \le 1 \end{cases}.$$

*The estimate is sharp for $|c_1| = 0$, and for $\frac{1}{1+3/4|\mu|} \le |c_1| \le 1$ when $\mu$ is nonnegative real number. The extremal functions are $\omega_1$ and $\omega_2$, respectively ($\omega_1$ and $\omega_2$ as defined in the proof of Theorem 2.1).*

For $\mu = 2$ in Theorem 2.2 we receive

**Corollary 2.1.** *If $\omega \in \mathcal{B}_0'$ is of form* (1.1). *Then*

$$|c_3 - 2c_1c_2| \le \begin{cases} \frac{1}{12}(1+|c_1|)\left[9|c_1|^2 - 4|c_1| + 4\right], & 0 \le |c_1| \le \frac{2}{5} \\ \frac{4}{3}|c_1|(1-|c_1|^2), & \frac{2}{5} \le |c_1| \le 1 \end{cases}.$$

*The estimate is sharp for $|c_1| = 0$ and for $\frac{2}{5} \le |c_1| \le 1$, with extremal functions $\omega_1$ and $\omega_2$, respectively ($\omega_1$ and $\omega_2$ as defined in the proof of Theorem 2.1).*

Next, for the modulus $|c_1c_3 - c_2^2|$ we have the following sharp estimate.

**Theorem 2.3.** *If $\omega \in \mathcal{B}_0'$ is of form* (1.1)*, then the following estimate is sharp*

$$|c_1c_3 - c_2^2| \le \frac{1}{42}(1-|c_1|^2)(3+|c_1|^2), \quad 0 \le |c_1| \le 1. \tag{2.3}$$

*Proof.* Using Lemma 1.2 we have

$$\begin{aligned} |c_1c_3 - c_2^2| &\le |c_1||c_3| + |c_2|^2 \\ &\le |c_1| \cdot \frac{1}{3}\left(1 - |c_1|^2 - \frac{4|c_2|^2}{1+|c_1|}\right) + |c_2|^2 \\ &= \frac{1}{3}|c_1|(1-|c_1|^2) + |c_2|^2 \cdot \frac{3-|c_1|}{3(1+|c_1|)} \\ &\le \frac{1}{3}|c_1|(1-|c_1|^2) + \frac{3-|c_1|}{3(1+|c_1|)} \cdot \frac{1}{4}(1-|c_1|^2)^2 \\ &= \frac{1}{12}(1-|c_1|^2)(2+|c_1|^2). \end{aligned}$$

The equality in (2.3) is obtained for the function $\omega_2(z)$ given in Theorem 2.1,

$$\omega_2(z) = \int_0^z \frac{|c_1| + z}{1+|c_1|z}\, dz = |c_1|z + \frac{1}{2}(1-|c_1|^2)z^2 - \frac{1}{3}|c_1|(1-|c_1|^2)z^3 + \cdots .$$

□

**Remark 2.2.** *From* (2.3) *we have taht for every $0 \le |c_1| \le 1$,*

$$|c_1c_3 - c_2^2| \le \frac{1}{12}\left(3 - 2|c_1|^2 - |c_1|^4\right) \le \frac{1}{4}.$$

**Remark 2.3.** *As it is shown in [2], for the class $\mathcal{U}$ of functions $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ defined by the condition*

$$\left|\left(\frac{z}{f(z)}\right)^2 f'(z) - 1\right| < 1, \qquad z \in \mathbb{D},$$

*we have*

$$\frac{z}{f(z)} = 1 - a_2 z - z\omega(z), \tag{2.4}$$

*where* $\omega \in \mathcal{B}_0'$ *and* $\omega(z) = c_1 z + c_2 z^2 + \cdots$. *From* (2.4) *we can express the coefficients* $a_3$, $a_4$, *and* $a_5$, *of the function* $f$, *depending on* $a_2$, $c_1$, $c_2$,, $c_3$,.... *After some calculations we receive*

$$|H_3(1)(f)| = \left|c_1 c_3 - c_2^2\right| \le \frac{1}{4},$$

*where* $H_3(1)(f)$ *is the Hankel determinant of third order (see [3]) and that result is the best possible. This property was the inspiration to study the class* $\mathcal{B}_0'$ *as a continuation of the study of the class* $\mathcal{B}_0$ *in [4].*

Similarly as in Theorem 2.2 we get

**Theorem 2.4.** *If* $\omega \in \mathcal{B}_0'$ *is of form* (1.1) *and* $\mu \in \mathcal{C}$, *then*

(2.5)

$$|c_1 c_3 - \mu c_2^2| \le \begin{cases} \frac{1}{3}|c_1|(1 - |c_1|^2), & |\mu| \le \frac{4}{3}\frac{|c_1|}{1+|c_1|} \\ \frac{1}{12}\left[3|\mu| + 2(2 - 3|\mu|)|c_1|^2 - (4 - 3|\mu|)|c_1|^4\right], & |\mu| \ge \frac{4}{3}\frac{|c_1|}{1+|c_1|} \end{cases}.$$

Finally, for the modulus $|c_4 - c_2^2|$ we have

**Theorem 2.5.** *If* $\omega \in \mathcal{B}_0'$ *is of form* (1.1), *then*

$$|c_4 - c_2^2| \le \frac{1}{4}(1 - |c_1|^2) \tag{2.6}$$

*and the estimate is sharp as the function*

$$\omega(z) = \int_0^z \frac{|c_1| + z^3}{1 + |c_1|z^3}\, dz = |c_1|z + \frac{1}{4}(1 - |c_1|^2)z^4 - \frac{1}{6}|c_1|(1 - |c_1|^2)z^6 + \cdots.$$

*shows.*

*Proof.* Using Lemma 1.2, we easily get

$$|c_4 - c_2^2| \le |c_4| + |c_2|^2 \le \frac{1}{4}(1 - |c_1|^2 - 4|c_2|^2) + |c_2|^2 = \frac{1}{4}(1 - |c_1|^2).$$

□

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