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UNIVALENCY OF CERTAIN TRANSFORM OF UNIVALENT FUNCTIONS

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Abstract

We consider univalency problem in the unit disc $\mathbb{D}$ of the function

$$g(z) = \frac{(z/f(z)) - 1}{-a_2},$$

where f belongs to some classes of univalent functions in $\mathbb{D}$ and $a_2 = \frac{f''(0)}{2} \neq 0$.
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1. Introduction. Let $\mathcal{A}$ denote the family of all analytic functions f in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ satisfying the normalization $f(0) = 0 = f'(0) - 1$, i.e., f has the form

$$f(z) = z + a_2z^2 + a_3z^3 + \dots . \tag{1}$$

Let $\mathcal{S}$, $\mathcal{S} \subset \mathcal{A}$, denote the class of univalent functions in $\mathbb{D}$, let $\mathcal{S}^\star$ be the subclass of $\mathcal{A}$ (and $\mathcal{S}$ which are starlike in $\mathbb{D}$) and let $\mathcal{U}$ denote the set of all $f \in \mathcal{A}$ satisfying the condition

$$|\mathrm{U}_f(z)| < 1 \qquad (z \in \mathbb{D}), \tag{2}$$

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where

$$\text{(3)} \qquad \mathrm{U}_f(z) := \left(\frac{z}{f(z)}\right)^2 f'(z) - 1.$$

In [1], Theorem 4 the authors consider the problem of univalency for the function

$$\text{(4)} \qquad g(z) = \frac{(z/f(z)) - 1}{-a_2},$$

where $f \in \mathcal{U}$ has the form (1) with $a_2 \neq 0$. They proved the following

Theorem A. *Let $f \in \mathcal{U}$. Then, for the function g defined by expression* (4) *we have*

(a) $|g'(z) - 1| < 1$ *for* $|z| < |a_2|/2$;

(b) $g \in \mathcal{S}^\star$ *in the disk* $|z| < |a_2|/2$, *and even more*

$$\left|\frac{zg'(z)}{g(z)} - 1\right| < 1$$

in the same disk;

(c) $g \in \mathcal{U}$ *in the disk* $|z| < |a_2|/2$ *if* $0 < |a_2| \leq 1$.

These results are the best possible.

For the proof of the previous theorem the authors used the next representation for the class $\mathcal{U}$ (see [2] and [3]). Namely, if $f \in \mathcal{U}$, then

$$\text{(5)} \qquad \frac{z}{f(z)} = 1 - a_2 z - z\omega(z),$$

where the function ω is analytic in $\mathbb{D}$ with $|\omega(z)| \leq |z| < 1$ for all $z \in \mathbb{D}$. The appropriate function g from (4) has the form

$$\text{(6)} \qquad g(z) = z + \frac{1}{a_2} z\omega(z).$$

2. Results. In this paper we consider other cases of Theorem A(c) and certain related results.

Theorem 1. *Let $f \in \mathcal{U}$. Then the function g defined by equation* (4) *belongs to $\mathcal{U}$ in the disc*

$$|z| < \sqrt{\frac{1 - |a_2| + \sqrt{|a_2|^2 + 2|a_2| - 3}}{2}},$$

i.e., satisfies (2) *on this disc, if* $\frac{5}{4} \leq |a_2| \leq 2$.

Proof. For the first part of the proof we use the same method as in [1]. By the definition of the class $\mathcal{U}$, i.e., inequality (2), and using the next estimation for the function ω

$$|z\omega'(z) - \omega(z)| \le \frac{r^2 - |\omega(z)|^2}{1 - r^2},$$

where $|z| = r$ and $|\omega(z)| \le r$, after some calculations we obtain

$$|\,\mathrm{U}_g(z)| = \left| \frac{\frac{1}{a_2}\left[z\omega'(z) - \omega(z)\right] - \frac{1}{a_2^2}\omega^2(z)}{\left[1 + \frac{1}{a_2}\omega_1(z)\right]^2} \right| \le \frac{|a_2| \cdot |z\omega'(z) - \omega(z)| + |\omega(z)|^2}{\left(|a_2| - |\omega(z)|\right)^2}$$

$$\le \frac{|a_2| \cdot \frac{r^2 - |\omega(z)|^2}{1 - r^2} + |\omega(z)|^2}{\left(|a_2| - |\omega(z)|\right)^2} =: \frac{1}{1 - r^2} \cdot \varphi(t).$$

Here,

$$\varphi(t) = \frac{|a_2| r^2 - (|a_2| - 1 + r^2) t^2}{(|a_2| - t)^2} \tag{7}$$

and $|\omega(z)| = t$, $0 \le t \le r$. From here we have that

$$\varphi'(t) = \frac{2|a_2|}{(|a_2| - t)^3} \cdot \left[r^2 - (|a_2| - 1 + r^2)t\right],$$

(where $|a_2| - t > 0$ since $|a_2| \ge \frac{5}{4} > 1 > t$). Next, $\varphi'(t) = 0$ for

$$t_0 = \frac{r^2}{|a_2| - 1 + r^2}$$

and $0 \le t_0 \le r$ if

$$\frac{r^2}{|a_2| - 1 + r^2} \le r,$$

which is equivalent to

$$r^2 - r + |a_2| - 1 \ge 0.$$

The last relation is valid for $\frac{5}{4} \le |a_2| \le 2$ and every $0 \le t < 1$. It means that the maximal value of the function φ on $[0, r]$ is

$$\varphi(t_0) = \frac{(|a_2| - 1 + r^2) r^2}{(|a_2| - 1)(|a_2| + r^2)}.$$

Finally,

$$|\,\mathrm{U}_g(z)| \le \frac{1}{1 - r^2} \cdot \varphi(t_0) = \frac{(|a_2| - 1 + r^2) r^2}{(1 - r^2)(|a_2| - 1)(|a_2| + r^2)} < 1$$

if

$$r^4 - (1-|a_2|)r^2 + (1-|a_2|) < 0,$$

or if

$$r < \sqrt{\frac{1-|a_2| + \sqrt{|a_2|^2 + 2|a_2| - 3}}{2}}.$$

This completes the proof. □

For our next consideration we need the following lemma.

Lemma 1. *Let $f \in \mathcal{A}$ be of the form* (1). *If*

$$\sum_2^\infty n|a_n| \le 1, \tag{8}$$

then

$$|f'(z) - 1| < 1 \qquad (z \in \mathbb{D}),$$

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 \qquad (z \in \mathbb{D})$$

(i.e. $f \in \mathcal{S}^\star$), and $f \in \mathcal{U}$.

For the proof of $f \in \mathcal{U}$ in the lemma see [3], while the rest easily follows.

Further, let $\mathcal{S}^+$ denote the class of univalent functions in the unit disc with the representation

$$\frac{z}{f(z)} = 1 + b_1 z + b_2 z^2 + \cdots, \quad b_n \ge 0, \quad n = 1, 2, 3, \ldots. \tag{9}$$

For example, the Silverman class (the class with negative coefficients) is included in the class $\mathcal{S}^+$, as well as the Koebe function $k(z) = \frac{z}{(1+z)^2} \in \mathcal{S}^+$. The next characterization is valid for the class $\mathcal{S}^+$ (for details see [4])

$$f \in \mathcal{S}^+ \quad \Leftrightarrow \quad \sum_{n=2}^\infty (n-1) b_n \le 1. \tag{10}$$

Theorem 2. *Let $f \in \mathcal{S}^+$. Then the function g defined by* (4) *belongs to the class $\mathcal{U}$ in the disc $|z| < |a_2|/2$ and the result is the best possible.*

Proof. Using the representation (9), the corresponding function g has the form

$$g(z) = \frac{\frac{z}{f(z)} - 1}{-a_2} = \frac{\frac{z}{f(z)} - 1}{b_1} = z + \sum_2^\infty \frac{b_n}{b_1} z^n \quad (b_1 \ne 0),$$

and from here

$$\frac{1}{r} g(rz) = z + \sum_2^\infty \frac{b_n}{b_1} r^{n-1} z^n \quad (0 < r \le 1).$$

Then, after applying Lemma 1, we have

$$\sum_2^\infty n|a_n| = \sum_2^\infty n\frac{b_n}{b_1}r^{n-1} = \frac{1}{b_1}\sum_2^\infty (n-1)b_n\frac{n}{n-1}r^{n-1}$$
$$\leq \frac{2r}{b_1}\sum_2^\infty (n-1)b_n \leq \frac{2r}{b_1} \leq 1$$

if $r \leq b_1/2 = |a_2|/2$. It means, by the same lemma, that $g \in \mathcal{U}$ in the disc $|z| < |a_2|/2$.

In order to show that the result is the best possible, let us consider the function f_1 defined by

$$\frac{z}{f_1(z)} = 1 + bz + z^2, \quad 0 < b \leq 2. \tag{11}$$

Then, $f_1 \in \mathcal{S}^+$ is of type $f_1(z) = z - bz^2 + \cdots$, so the function

$$g_1(z) = \frac{\frac{z}{f_1(z)} - 1}{b} = z + \frac{1}{b}z^2$$

is such that

$$\left|\left(\frac{z}{g_1(z)}\right)^2 g_1'(z) - 1\right| \leq \frac{\frac{1}{b^2}|z|^2}{\left(1 - \frac{1}{b}|z|\right)^2} < 1$$

when $|z| < b/2$. This implies that g_1 belongs to the class $\mathcal{U}$ in the disc $|z| < b/2$. On the other hand, since $g_1'(-b/2) = 0$, the function g_1 is not univalent in a bigger disc, implying that the result is the best possible. □

Theorem 3. *Let $f \in \mathcal{S}$. Then the function g defined by* (4) *belongs to the class $\mathcal{U}$ in the disc $|z| < r_0$, where r_0 is the unique real root of equation*

$$\frac{3r^2 - 2r^4}{(1-r^2)^2} - \ln(1-r^2) = |a_2|^2 \tag{12}$$

on the interval $(0,1)$.

Proof. We apply the same method as in the proof of the previous theorem. Namely, if $f \in \mathcal{S}$ has the representation (9), then

$$\sum_{n=2}^\infty (n-1)|b_n|^2 \leq 1 \tag{13}$$

(see [5], Theorem 11, p. 193, Vol. 2). Also, using (4), (9) and (13), we have $a_2 = -b_1$, and

$$\frac{1}{r}g(rz) = z + \sum_2^\infty \frac{b_n}{b_1}r^{n-1}z^n, \quad 0 < r \leq 1.$$

So,

$$\begin{aligned}
\sum_{n=2}^{\infty} n|a_n| &= \sum_{n=2}^{\infty} n\frac{|b_n|}{|b_1|}r^{n-1} \\
&= \frac{1}{|b_1|}\sum_{n=2}^{\infty}\sqrt{n-1}\cdot|b_n|\cdot\frac{n}{\sqrt{n-1}}\cdot r^{n-1} \\
&\le \frac{1}{|b_1|}\cdot\left(\sum_{n=2}^{\infty}(n-1)|b_n|^2\right)^{1/2}\cdot\left(\sum_{n=2}^{\infty}\frac{n^2}{n-1}r^{2(n-1)}\right)^{1/2} \\
&\le \frac{1}{|b_1|}\left(r^2\sum_{n=2}^{\infty}(n-1)(r^2)^{n-2}+2r^2\sum_{n=2}^{\infty}(r^2)^{n-2}+\sum_{n=2}^{\infty}\frac{1}{n-1}(r^2)^{n-1}\right)^{1/2} \\
&= \frac{1}{|b_1|}\left[\frac{3r^2-2r^4}{(1-r^2)^2}-\ln(1-r^2)\right]^{1/2}\le 1
\end{aligned}$$

if $|z|<r_0$, where r_0 is the root of the equation

$$\frac{3r^2-2r^4}{(1-r^2)^2}-\ln(1-r^2)=|b_1|^2\ (=|a_2|^2).$$

We note that the function on the left side of this equation is an increasing one on the interval $(0,1)$, so the equation has a unique root when $0<|a_2|\le 2$. $\square$

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