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# On certain properties of some subclasses of univalent functions

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**Abstract.** In this paper we determine the disks $|z| < r \leq 1$ where for different classes of univalent functions, we have the property

$$\operatorname{Re}\left\{2\frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)}\right\} > 0 \qquad (|z| < r).$$

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Let $\mathcal{A}$ denote the family of all analytic functions in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ satisfying the normalization $f(0) = 0 = f'(0) - 1$.

Further, let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of all univalent functions in $\mathbb{D}$, and $\mathcal{S}^\star$ and $\mathcal{K}$ be the subclasses of $\mathcal{A}$ of functions that are starlike and convex in $\mathbb{D}$, respectively. Next, let $\mathcal{U}$ denote the set of all $f \in \mathcal{A}$ satisfying the condition

$$\left|\left(\frac{z}{f(z)}\right)^2 f'(z) - 1\right| < 1 \qquad (z \in \mathbb{D}).$$

More on this class can be found in [4, 5, 9].

Next, by $\mathcal{G}$ we denote the class of all $f \in \mathcal{A}$ in $\mathbb{D}$ satisfying the condition

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < \frac{3}{2} \qquad (z \in \mathbb{D}).$$

More about the class $\mathcal{G}$ one can find in [2] and [7].

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In their paper ([3]) Miller and Mocanu introduced the classes of $\alpha$-convex functions $f \in \mathcal{A}$ by the next condition:

$$\operatorname{Re}\left\{(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}>0 \qquad (z\in\mathbb{D}), \tag{1}$$

where $\frac{f(z)f'(z)}{z} \neq 0$ for all $z \in \mathbb{D}$, and $\alpha \in \mathbb{R}$. Those classes they denoted by $\mathcal{M}_\alpha$ and proved the next

**Theorem A.**
*(a)* $\mathcal{M}_\alpha \subseteq \mathcal{S}^\star$ *for every* $\alpha \in \mathbb{R}$*;*
*(b)* $\mathcal{M}_1 = \mathcal{K} \subseteq \mathcal{M}_\alpha \subseteq \mathcal{S}^\star$ *for* $0 \le \alpha \le 1$*;*
*(c)* $\mathcal{M}_\alpha \subset \mathcal{M}_1 = \mathcal{K}$ *for* $\alpha > 1$**.**

In [8] the authors proved
**Theorem B.**
*(a)* $\mathcal{M}_\alpha \subset \mathcal{U}$ *for* $\alpha \le -1$*;*
*(b)* $\mathcal{M}_\alpha$ *is not subset of* $\mathcal{U}$ *for* $0 \le \alpha \le 1$.

Choosing $\alpha = -1$ in Theorem A(a) and Theorem B(a), from (1), we have that the condition

$$\operatorname{Re}\left\{2\frac{zf'(z)}{f(z)}-\frac{zf''(z)}{f'(z)}\right\}>1 \qquad (z\in\mathbb{D}) \tag{2}$$

implies $f \in \mathcal{S}^\star \cap \mathcal{U}$, i.e., the above inequality is sufficient for univalence in the unit disc. As expected, it is not necessary condition for univalence, i.e., univalent functions does not necessarily have property (2). See functions $f_2$ and $f_3$ analysed bellow.

But, is the following weaker inequality necessary for univalence

$$\operatorname{Re}\left\{2\frac{zf'(z)}{f(z)}-\frac{zf''(z)}{f'(z)}\right\}>0 \qquad (z\in\mathbb{D})?$$

The answer is also negative. Even more, it is not necessary condition even for starlikeness, nor for the classes $\mathcal{U}$ and $\mathcal{G}$.

Namely, let consider the differential operator

$$D(f;z) := 2\frac{zf'(z)}{f(z)}-\frac{zf''(z)}{f'(z)} \tag{3}$$

and the functions

$$k(z)=\frac{z}{(1-z)^2},\ f_1(z)=\frac{z}{1-z^2},\ f_2(z)=-\log(1-z),$$

and

$$f_3(z)=\frac{z(1-\frac{1}{\sqrt{2}}z)}{1-z^2}.$$

Then, we have, respectively,

$$
\begin{aligned}
D(k;z) &= 1 + \frac{1+z^2}{1-z^2},\\
D(f_1;z) &= 1 + \frac{1-z^2}{1+z^2},\\
D(f_2;z) &= -\frac{z(2+\log(1-z))}{(1-z)\log(1-z)},\\
D(f_3;z) &= \frac{-\sqrt{2}z^3+3z^2-3\sqrt{2}z+2}{(1-\frac{1}{\sqrt{2}}z)(1-\sqrt{2}z+z^2)}.
\end{aligned}
$$

From the previous remark we easily conclude that the functions $k$ and $f_1$ belong to the class $\mathcal{S}^\star \cap \mathcal{U}$, but for the function $f_2$ (which is convex) for $z = r$, $0 \le r < 1$, we have

$$D(f_2;r) = -\frac{r(2+\log(1-r))}{(1-r)\log(1-r)} < 0$$

if $1 - e^{-2} = 0.86466\ldots \le r < 1$. Also, we note that $f_2 \notin \mathcal{U}$.

For the function $f_3$, in [6], the authors showed that it is close-to-convex and univalent in $\mathbb{D}$, but not in $\mathcal{U}$. Additionally, $\operatorname{Re}[D(f;z)] > 0$ does not hold on the unit disk. Indeed, let we put

$$D(f_3;z) =: \frac{g(z)}{h(z)}, \tag{4}$$

where

$$g(z) = -\sqrt{2}z^3 + 3z^2 - 3\sqrt{2}z + 2$$

and

$$h(z) = \left(1 - \frac{1}{\sqrt{2}}z\right)(1-\sqrt{2}z+z^2),$$

and use $z = r$, $0 \le r < 1$. Then it is evident that $h(r) > 0$ for all $0 \le r < 1$. Also we have $g'(r) = -3(\sqrt{2}r^2 - 2r + \sqrt{2}) < 0$ for all $r \in [0,1)$, which implies that $g(r)$ is a decreasing function on the interval $[0,1)$. Thus, $2 = g(0) \ge g(r) > g(1/\sqrt{2}) = 0$ for $0 \le r < 1/\sqrt{2}$, and $g(r) \le 0$ for $\frac{1}{\sqrt{2}} \le r < 1$. Now, from (4), we easily conclude that the condition $\operatorname{Re}[D(f;z)] > 0$ is not satisfied for the function $f_3$ in the disc $|z| < r$, where $\frac{1}{\sqrt{2}} \le r < 1$, i.e., for close-to-convex functions, $\operatorname{Re}[D(f;z)] > 0$ on a disk with radius smaller then $\frac{1}{\sqrt{2}} = 0.7071\ldots$.

The above analysis raises the question of finding radius $r_*$ for each of the classes defined above, such that $\operatorname{Re}[D(f;z)] > 0$ at least in the disc $|z| < r_*$. The next theorem answers this question. We don't know if the values for $r_*$ are the best possible.

**Theorem 1.** *Let $D(f;z)$ be defined by* (3)*. Then*

$$\operatorname{Re}[D(f;z)] > 0 \qquad (|z| < r_*)$$

*in each of the following cases:*

(*i*) *$f \in \mathcal{U}$ and $r_* = r_1 = 0.839\ldots$ is the root of the equation $r^3 + 2r^2 - 2 = 0$;*

(*ii*) $f \in \mathcal{S}^\star(1/2)$ *and* $r_* = r_2 = \sqrt{\frac{\sqrt{5}-1}{2}} = 078615\ldots$;
(*iii*) $f \in \mathcal{G}$ *and* $r_* = r_3 = \frac{2}{3} = 0.666\ldots$;
(*iv*) $f \in \mathcal{S}^\star$ *and* $r_* = r_4 = \frac{1}{2} = 0.5$;
(*v*) $f \in \mathcal{S}$ *and* $r_* = r_5 = \frac{1}{4} = 0.25$.

*Proof.* (i) First, from the definition of the class $\mathcal{U}$, we easily conclude that $f \in \mathcal{U}$ if, and only if, there exists a function $\phi$, analytic in $\mathbb{D}$ with $|\phi(z)| \le 1$ in $\mathbb{D}$, such that

$$\left[\frac{z}{f(z)}\right]^2 f'(z) = 1 + z^2\phi(z). \tag{5}$$

From (5), after some calculations, we obtain that

$$2\frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)} = 2\frac{1 - \frac{1}{2}z^3\phi'(z)}{1 + z^2\phi(z)}. \tag{6}$$

Since $|\phi(z)| \le 1$, then

$$|\phi'(z)| \le \frac{1 - |\phi(z)|^2}{1 - |z|^2}, \tag{7}$$

(see [1, p.198]) and from here

$$\left|\frac{1}{2}z^3\phi'(z)\right| \le \frac{|z^3|}{2(1-|z|^2)}(1 - |\phi(z)|^2) < 1 - |\phi(z)|^2, \tag{8}$$

because $\frac{|z^3|}{2(1-|z|^2)} < 1$ for $|z| < r_1$. Also,

$$|z^2\phi(z)| < |r_1^2\phi(z)| < \frac{1}{\sqrt{2}}|\phi(z)|, \tag{9}$$

since

$$r_1^2 = 0.7044\ldots < \frac{1}{\sqrt{2}} = 0.7071\ldots.$$

Finally, by using (7),(8) and (9), we have

$$\begin{aligned}\left|\arg\left[2\frac{1 - \frac{1}{2}z^3\phi'(z)}{1 + z^2\phi(z)}\right]\right| &\le |\arg\left[1 - \frac{1}{2}z^3\phi'(z)\right]| + |\arg(1 + z^2\phi(z))| \\ &< \arcsin(1 - |\phi(z)|^2) + \arcsin\left(\frac{1}{\sqrt{2}}|\phi(z)|\right) \\ &= \arcsin\sqrt{1 - \frac{1}{2}|\phi(z)|^2} \\ &\le \arcsin 1 \\ &= \frac{\pi}{2},\end{aligned}$$

which implies $\mathrm{Re}[D(f;z)] > 0$.

(ii) Since $f \in \mathcal{S}^\star(1/2)$, we can put

$$\frac{zf'(z)}{f(z)} = \frac{1}{1 - \omega(z)},$$

where $\omega$ is analytic in $\mathbb{D}$, $\omega(0)=0$ and $|\omega(z)|<1$ for all $z\in\mathbb{D}$. From here we have that

$$\frac{zf''(z)}{f'(z)}=\frac{z\omega'(z)}{1-\omega(z)}+\frac{1}{1-\omega(z)}-1,$$

and so

$$D(f;z)=2-\frac{z\omega'(z)-\omega(z)}{1-\omega(z)}.$$

Since $\omega(0)=0$ and $|\omega(z)|<1$, $z\in\mathbb{D}$, implies that $\left|\frac{\omega(z)}{z}\right|\le 1$, $z\in\mathbb{D}$, then by using the estimate (7) (with $\frac{\omega(z)}{z}$ in stead of $\phi$), we obtain

$$|z\omega'(z)-\omega(z)|\le\frac{r^2-|\omega(z)|^2}{1-r^2}, \tag{10}$$

(where $|z|=r$ and $|\omega(z)|\le r$). Further, we have

$$\begin{aligned}\mathrm{Re}[D(f;z)]&\ge 2-\frac{|z\omega'(z)-\omega(z)|}{1-|\omega(z)|}\\&\ge 2-\frac{1}{1-r^2}\frac{r^2-|\omega(z)|^2}{1-|\omega(z)|}\\&=2-\frac{1}{1-r^2}\varphi(t),\end{aligned}$$

where we put $|\omega(z)|=t$, $0\le t\le r$ and $\varphi(t)=\frac{r^2-t^2}{1-t}$. By elementary calculation we obtain that $\varphi(t)\le 2(1-\sqrt{1-r^2})$ for $t\in[0,r]$. This implies that

$$\mathrm{Re}[D(f;z)]\ge 2-\frac{2(1-\sqrt{1-r^2})}{1-r^2}=2\frac{\sqrt{1-r^2}-r^2}{1-r^2}>0,$$

since $|z|=r<\sqrt{\frac{\sqrt{5}-1}{2}}=r_2$.

(iii) For $f\in\mathcal{G}$ in [2] is proven that

$$\frac{zf'(z)}{f(z)}\prec\frac{1-z}{1-\frac{z}{2}},$$

i.e., that

$$\frac{zf'(z)}{f(z)}=\frac{1-\omega(z)}{1-\frac{\omega(z)}{2}},$$

where $\omega$ is analytic in $\mathbb{D}$ such that $\omega(0)=0$ and $|\omega(z)|<1$ for $z\in\mathbb{D}$. From the last relation we easily obtain

$$D(f;z)=2-\frac{\omega(z)}{2-\omega(z)}+\frac{z}{(1-\omega(z))(2-\omega(z))}\omega'(z)$$

and from here

$$\mathrm{Re}[D(f;z)]\ge 2-\frac{|\omega(z)|}{2-|\omega(z)|}-\frac{|z|}{(1-|\omega(z)|)(2-|\omega(z)|)}|\omega'(z)|.$$

Applying the inequality (7), we give

$$\mathrm{Re}[D(f;z)] \ge 2 - \frac{|\omega(z)|}{2-|\omega(z)|} - \frac{|z|}{(1-|\omega(z)|)(2-|\omega(z)|)}\frac{1-|\omega(z)|^2}{1-|z|^2},$$

or, if we use $|\omega(z)| \le r$, where $|z| = r$:

$$\mathrm{Re}[D(f;z)] \ge 2 - \frac{r}{1-r} = \frac{2-3r}{1-r} > 0,$$

since $r < \frac{2}{3} = r_3$.

(iv) We can use relation (7) and the same method as in the previous cases. Namely, now we can put

$$\frac{zf'(z)}{f(z)} = \frac{1+\omega(z)}{1-\omega(z)},$$

where $\omega$ is analytic in $\mathbb{D}$, $\omega(0) = 0$ and $|\omega(z)| < 1$ for $z \in \mathbb{D}$. Then,

$$\frac{zf''(z)}{f'(z)} = 2\frac{z\omega'(z)}{1-\omega^2(z)} + \frac{1+\omega(z)}{1-\omega(z)} - 1$$

and after that

$$D(f;z) = \frac{2}{1-\omega(z)} - 2\frac{z\omega'(z)}{1-\omega^2(z)}.$$

Finally, we have (using $|\omega(z)| \le r$, where $|z| = r$):

$$\begin{aligned}
\mathrm{Re}[D(f;z)] &\ge \mathrm{Re}\frac{2}{1-\omega(z)} - 2\frac{|z||\omega'(z)|}{1-|\omega(z)|^2} \\
&\ge \frac{2}{1+|\omega(z)|} - 2\frac{|z|}{1-|\omega(z)|^2}\frac{1-|\omega(z)|^2}{1-|z|^2} \\
&\ge \frac{2}{1+r} - \frac{2r}{1-r^2} \\
&= 2\frac{1-2r}{1-r^2} > 0,
\end{aligned}$$

if $|z| = r < \frac{1}{2} = r_4$.

(v) If $f \in \mathcal{S}$ then, from the classical result (see [1, p.32]), we have

$$\left|\log\frac{zf'(z)}{f(z)}\right| \le \log\frac{1+r}{1-r}, \quad |z| = r < 1.$$

If we put $w = \log\frac{zf'(z)}{f(z)}$ and $R = \log\frac{1+r}{1-r}$, then we have $\frac{zf'(z)}{f(z)} = e^w$, where $|w| \le R$. If we choose $r \le \tanh\frac{1}{2} = \frac{e-1}{e+1} = 0.46...$, then we have $R \le 1$. For such $R$ the function $e^w$ is convex with positive real coefficients that maps the unit disk onto a region that is symmetric with respect to the real axes ($\overline{e^w} = e^{\overline{w}}$), with diameter end points for $w = -1$ and $w = 1$. This implies that

$$\mathrm{Re}\frac{zf'(z)}{f(z)} = \mathrm{Re}(e^w) \ge e^{-R} = \frac{1-r}{1+r}.$$

Also, from the relation for functions from the class $\mathcal{S}$ (see [1, Theorem 2.4, p.32]) we have

$$\left|\frac{zf''(z)}{f'(z)}-\frac{2r^2}{1-r^2}\right|\leq\frac{4r}{1-r^2}$$

and from here

$$\mathrm{Re}\frac{zf''(z)}{f'(z)}\leq\frac{2r^2}{1-r^2}+\frac{4r}{1-r^2}=2\frac{2r+r^2}{1-r^2}.$$

Finally,

$$\mathrm{Re}[D(f;z)]\geq 2\frac{1-r}{1+r}-2\frac{2r+r^2}{1-r^2}=2\frac{1-4r}{1-r^2}>0$$

if $|z|=r<\frac{1}{4}=r_5$. □

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