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Univalence and Starlikeness of Certain Classes of Analytic Functions

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Abstract: For the analytic functions $\phi(\zeta) = \zeta + \sum_{k=n}^{\infty} \phi_k \zeta^k$ in the unit disk $\mathbb{O}$, we calculate the values of n and α, where the condition $\Re(1 + \zeta\phi''(\zeta)/\phi'(\zeta)) > -\alpha$ or $\Re(1 + \zeta\phi''(\zeta)/\phi'(\zeta)) < 1 + \alpha/2$ yields univalence and starlikeness. Conditions imply ϕ in the class where all normalized analytic functions $\mathcal{U}$, with $\left|(\zeta/\phi(\zeta))^2\phi'(\zeta) - 1\right| < 1$ are obtained. Recent findings are gained, and unique cases are demonstrated. The generalization of the Jack lemma serves the proof of the main result and that our methodology is based on the idea of subordination.

Keywords: analytic function; subordination; starlike function; univalent function; Jack lemma; open unit disk; starlike function; convex function

1. Introduction

Symmetry in the open unit disk $\mathbb{O} := \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ can refer to several different types of symmetry, including rotational symmetry, reflection symmetry, and inversion symmetry. Inversion symmetry refers to the property that the open unit disk looks the same when inverted with respect to a certain point. The open unit disk has inversion symmetry with respect to its center (the origin), because inverting any complex number ζ in the disk with respect to the origin gives the complex number $1/\zeta$, which is also in the disk. Generally, the open unit disk has a rich set of symmetries, which can be useful in a variety of mathematical and geometric contexts. In this effort we aim to explore more geometric properties in this symmetry domain.

A function is said to be starlike if it maps a disk in the complex plane onto a shape that is itself star-shaped with respect to some fixed point in the disk. In other words, a function is starlike if its image under a suitable scaling and rotation is contained in a star-shaped domain, where the star-shaped domain is obtained by connecting the fixed point to all other points in the domain using straight line segments. Another term for a starlike function is a convex function. Both univalent functions and starlike functions are important subclasses of analytic functions in complex analysis, and they have many interesting properties and applications. For example, univalent functions are often used in geometric function theory to study conformal mappings and the Riemann mapping theorem, while starlike functions are used in geometric function theory and mathematical physics to model phenomena such as electrostatics [1,2] and fluid flow [3,4].

Ozaki [5] presented a condition on a normalized class of analytic function to univalently satisfy thw real inequality

$$\Re\left(1 + \frac{\zeta\phi''(\zeta)}{\phi'(\zeta)}\right) < \frac{3}{2}.$$

Consequently, the above inequality is used to show the convexity and close to convex properties in one direction [6,7], respectively. Currently, as an application of the Ozaki

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inequality, many classes of analytic functions involve polynomials, special functions and different types of operators of the normalized class are investigated. For example, the starlikeness property is studied in [8–10]; the convex property is checked in [11–13] and the close to convex property is realized in [14,15].

In this effort, we proceed with the investigation on univalency and starlikeness of the normalized class. We shall deal with the $\mathcal{U}$ class such that the inequality

$$\left| \left(\frac{\zeta}{\phi(\zeta)} \right)^2 \phi'(\zeta) - 1 \right| < 1$$

is satisfied. Some recent results are obtained and special cases are illustrated. Our methodology is based on the concept of subordination and the proof is given by the generalization of Jack lemma.

The effort is organized as follows: Section 2 deals with the general information that can be utilized in the proof. Section 3 shows our results and their consequences. Section 4 is the conclusion of this work.

2. Information

Let $\mathcal{A}_n, n \geq 2$ denote the set of all analytic functions ϕ in $\mathbb{O}$ taking the power series

$$\phi(\zeta) = \zeta + \phi_n \zeta^n + \phi_{n+1} \zeta^{n+1} + \cdots$$

with $\mathcal{A}_2 = \mathcal{A}$. Let $\mathcal{S}$ be its subclass of univalent functions and $\mathcal{S}^* \subset \mathcal{S}$ be the class of all starlike univalent functions. Every $\phi \in \mathcal{S}^*$ characterized analytically by

$$\text{Re}\left\{ \frac{\zeta \phi'(\zeta)}{\phi(\zeta)} \right\} > 0, \ \zeta \in \mathbb{O}.$$

A function $\phi \in \mathcal{S}$ is called convex $(\mathcal{C})$ if and only if $\zeta \phi' \in \mathcal{S}^*$ (see [16]). Let $\mathcal{U}$ be the set of all $\phi \in \mathcal{A}$ achieving the inequality $|\mathcal{U}_\phi(\zeta)| < 1$ for $\zeta \in \mathbb{O}$, where

$$\mathcal{U}_\phi(\zeta) = \left(\frac{\zeta}{\phi(\zeta)} \right)^2 \phi'(\zeta) - 1.$$

For example of functions in $\mathcal{U}_\phi(\zeta)$, one can realize that the following set:

$$S_{\mathbb{Z}} = \left\{ \zeta, \frac{\zeta}{(1 \pm \zeta)^2}, \frac{\zeta}{1 \pm \zeta}, \frac{\zeta}{1 \pm \zeta^2}, \frac{\zeta}{1 \pm \zeta + \zeta^2} \right\}$$

belongs to $\mathcal{U}$. Clearly,

$$\kappa(\zeta) = \frac{\zeta}{(1-\zeta)^2} \in \mathcal{U} \bigcap \mathcal{S}^*.$$

Moreover, it recognized the following inclusion [17]:

$$S_{\mathbb{Z}} \subset \mathcal{S}^* \subset \mathcal{S}.$$

Basic properties of the class $\mathcal{U}$ were studied in [18]. In recent years, the class $\mathcal{U}$ has received a lot of attention, for instance in the works of [19–37].

Recall that an analytic function ϕ is *subordinate* to the analytic function ψ symbolized by $\phi(\zeta) \prec \psi(\zeta)$), if there exists an analytic self-map ω of $\mathbb{O}$ with $\omega(0) = 0$ satisfying

$$\phi(\zeta) = \psi(\omega(\zeta)).$$

In the foregoing discussion, for $\phi \in \mathcal{A}_n, n \geq 2$ we determine values of n such that the condition

$$\mathrm{Re}\left(\frac{1+\zeta\phi''(\zeta)}{\phi'(\zeta)}\right) > -\alpha$$

or

$$\mathrm{Re}\left(\frac{1+\zeta\,\phi''(\zeta)}{\phi'(\zeta)}\right) < 1+\alpha/2$$

implies univalence and starlikeness. Additionally, conditions implying ϕ in the class $\mathcal{U}$ were obtained. For the analytic functions $\phi \in \mathcal{A}_2$ and $\alpha = 1$, the condition for starlikeness were determined for the class $\mathrm{Re}\,(1+\zeta\phi''(\zeta)/\phi'(\zeta)) < 3/2$ in [19].

We request the next result.

Lemma 1 ([20] (p. 19))**.** *Assume that* $\zeta_0 \in \mathbb{O}$ *and* $r_0 = |\zeta_0|$. *Moreover, assume that*

$$\phi(\zeta) = \phi_n\zeta^n + \phi_{n+1}\zeta^{n+1} + \cdots$$

is analytic on $\mathbb{O}_{r_0} \cup \{\zeta_0\}$ *with* $\phi(\zeta) \not\equiv 0$ *and* $n \geq 1$. *If*

$$|\phi(\zeta_0)| = \max\{|\phi(\zeta)| : \zeta \in \overline{\mathbb{O}}_{r_0}\},$$

then there exists an $m \geq n$ *such that*

- $\Re\left(\frac{\zeta_0\phi'(\zeta_0)}{\phi(\zeta_0)}\right) = m$, *and*
- $\Re\left(\frac{\zeta_0\phi''(\zeta_0)}{\phi'(\zeta_0)} + 1\right) \geq m.$

3. Main Results

In this section, we illustrate our main results. These results describe the univalency property via the behavior of bounded functions. Let r_0 be the largest radius such that $\phi(\zeta)$ maps the circle $|\zeta| = r_0$ inside the unit disk $\mathbb{O}$. Then, the bounded turning class of $\phi(\zeta)$ is the smallest non-negative number α such that $\phi(\zeta)$ maps the circle $|\zeta| = r$ inside the sector $|\arg\zeta| < \alpha$ for all r with $0 < r \leq r_0$.

In other words, the bounded turning class measures the maximum amount of turning that the function exhibits on the unit circle, as we move outward from the origin. The bounded turning class is an important concept in the theory of univalent functions, and it has been extensively studied in the literature. Properties of the bounded turning class are closely related to the geometric properties of univalent functions, such as the distortion theorem and the Koebe one-quarter theorem. For example, let

$$\phi(\zeta) = \frac{\zeta}{(1-\zeta)}, \quad \zeta \in \mathbb{O}$$

then the basic condition for bounded turning is that (see Figure 1)

$$\Re(\zeta/(1-\zeta)') > 0, \quad \zeta \in \mathbb{O}.$$

Theorem 1. *Let* $\alpha > 0$ *and* $\phi \in \mathcal{A}_n$ *satisfy the condition*

$$\mathrm{Re}\left(1+\frac{\zeta\phi''(\zeta)}{\phi'(\zeta)}\right) < 1+\frac{\alpha}{2}, \quad \zeta \in \mathbb{O} \tag{1}$$

(a) *If* $n \geq 1+\alpha$, *and* $\phi(\zeta)/\zeta \neq 0$, *then* $\mathrm{Re}\,(\phi'(\zeta)) > 0$, $\zeta \in \mathbb{O}$.
(b) *If* $n \geq 1+\alpha$, *and* $\phi(\zeta)/\zeta \neq 0$, *then*

$$\zeta\phi'(\zeta)/\phi(\zeta) \prec\ (1-\zeta)/(1-a\,\zeta),\ \zeta \in \mathbb{O}, \quad a = 1/(1+\alpha).$$

(c) *If* $n \geq 5 + \alpha$ *and* $\phi(\zeta)/\zeta \neq 0$, *then* $\phi \in \mathcal{U}$.

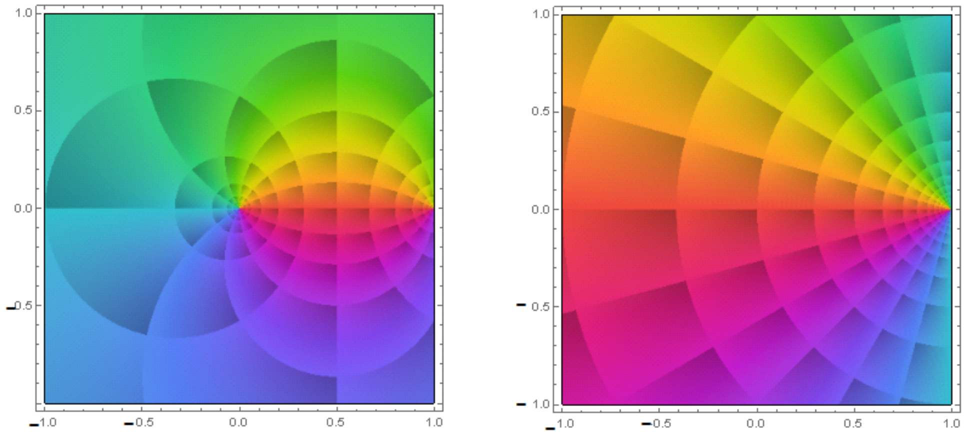

Figure 1. Plot of $\phi(\zeta) = \zeta/(1-\zeta)$ and its bounded turning behavior, respectively.

Proof.

(a) First, prove that $\phi'(\zeta) \neq 0$ for all $\zeta \in \mathbb{O}$ and $\zeta \neq 0$ (since $\phi'(0) = 1$). If there exists ζ_1, $0 < |\zeta_1| < 1$ and

$$\phi'(\zeta) = (\zeta - \zeta_1)^{m_1}\psi(\zeta),$$

where $m_1 \geq 1$ and ψ is analytic in $\mathbb{O}$ and $\psi(\zeta_1) \neq 0$, then

$$1 + \frac{\zeta\,\phi''(\zeta)}{\phi'(\zeta)} = 1 + \frac{m_1\zeta}{\zeta - \zeta_1} + \frac{\zeta\psi'(\zeta)}{\psi(\zeta)}.$$

Thus, $1 + \zeta\,\phi''(\zeta)/\phi'(\zeta) \to \infty$ when $\zeta \to \zeta_1$, which is a contradiction to (1).
Let

$$\phi'(\zeta) = (1 - \omega(\zeta))^{\frac{\alpha}{n-1}}, \tag{2}$$

where $0 < \alpha/(n-1) \leq 1$.
Since previous $\phi'(\zeta) \neq 0$ for all $\zeta \in \mathbb{O}$, then $\omega(\zeta) = b_{n-1}\zeta^{n-1} + \cdots$ is analytic in $\mathbb{O}$ with $\omega(0) = 0$. Furthermore, it follows from (2) that

$$\left(1 + \frac{\zeta\phi''(\zeta)}{\phi'(\zeta)}\right) = 1 - \frac{\alpha\zeta\omega'(\zeta)}{(n-1)(1-\omega(\zeta))}.$$

Consider that $\zeta_0 \in \mathbb{O}$, such that

$$\max_{|\zeta| \leq |\zeta_0|} |\omega(\zeta)| = |\omega(\zeta_0)| = 1.$$

In view of Lemma 1, we have

$$\zeta_0\omega'(\zeta_0) = k\omega(\zeta_0); \quad (k \geq n-1; \omega(\zeta_0) = e^{i\theta}; \theta \in \mathbb{R}),$$

thus

$$\mathrm{Re}\left(1 + \frac{\zeta_0\phi''(\zeta_0)}{\phi'(\zeta_0)}\right) = 1 - \frac{\alpha}{n-1}\mathrm{Re}\left(\frac{k\,e^{i\theta}}{1 - e^{i\theta}}\right).$$

Hence, the following is obtained

$$\mathrm{Re}\left(1 + \frac{\zeta_0\phi''(\zeta_0)}{\phi'(\zeta_0)}\right) = 1 + \frac{k\alpha}{2(n-1)} \geq 1 + \frac{\alpha}{2},$$

which is a contradiction to (1). It means that $|\omega(\zeta)| < 1, \zeta \in \mathbb{O}$, and from (2) finally leads to

$$|\arg(\phi'(\zeta))| = \frac{\alpha}{n-1}|\arg((1-\omega(\zeta))| < \frac{\alpha}{(n-1)}\frac{\pi}{2} \leq \frac{\pi}{2},$$

where $n \geq 1+\alpha$, which implies $\mathrm{Re}\,(\phi'(\zeta)) > 0$, $\zeta \in \mathbb{O}$,

(b) Define a function ω by

$$\frac{\zeta\phi'(\zeta)}{\phi(\zeta)} = \frac{1-\omega(\zeta)}{1-a\omega(\zeta)}, \tag{3}$$

where $a = 1/(1+\alpha)$, then $\omega(\zeta) = c_{n-1}\zeta^{n-1} + \cdots$ is analytic in $\mathbb{O}$. Additionally, suppose that there is a point $\zeta_0 \in \mathbb{O}$, such that

$$\max_{|\zeta|\leq|\zeta_0|} |\omega(\zeta)| = |\omega(\zeta_0)| = 1.$$

Then by applying Lemma 1, the following is acquired

$$\zeta_0\omega'(\zeta_0) = k\omega(\zeta_0); \quad (k \geq n-1; \omega(\zeta_0) = e^{i\theta}; \theta \in \mathbb{R}).$$

From (3), logarithmic differentiation, yields

$$\begin{aligned}
\mathrm{Re}\left(1+\frac{\zeta_0\phi''(\zeta_0)}{\phi'(\zeta_0)}\right) &= \mathrm{Re}\left(a\frac{\zeta_0\omega'(\zeta_0)}{1-a\omega(\zeta_0)} - \frac{\zeta_0\omega'(\zeta_0)}{1-\omega(\zeta_0)} + \frac{1-\omega(\zeta_0)}{1-a\omega(\zeta_0)}\right) \\
&= \mathrm{Re}\left(a\frac{k\,e^{i\theta}}{1-a\,e^{i\theta}} - \frac{k\,e^{i\theta}}{1-e^{i\theta}} + \frac{1-e^{i\theta}}{1-a\,e^{i\theta}}\right) \\
&= \frac{1+a}{2}\varphi(t), \quad t = cos\theta
\end{aligned}$$

where

$$\varphi(t) = \frac{k(1-a)+2(1-t)}{1-2at+a^2}, \; -1 \leq t \leq 1.$$

Since $\alpha > 0$, $0 < a = 1/(1+\alpha) < 1$ and $n \geq 1+\alpha$, then $na - 1 \geq 0$. Thus,

$$\begin{aligned}
\varphi'(t) &= 2(1-a)\frac{(k+1)a-1}{(1-2at+a^2)^2} \\
&\geq 2(1-a)\frac{na-1}{(1-2at+a^2)^2} \geq 0,
\end{aligned}$$

which implies that the function φ is a non-decreasing function and

$$\varphi(t) \geq \varphi(-1) = \frac{(n-1)(1-a)+4}{(1+a)^2}.$$

Hence, the previous relation shows that

$$\mathrm{Re}\left(1+\frac{\zeta_0\phi''(\zeta_0)}{\phi'(\zeta_0)}\right) \geq \frac{(n-1)(1-a)+4}{2(1+a)} \geq 1+\frac{\alpha}{2},$$

which is a contradiction to (1). From that $|\omega(\zeta)| < 1, \zeta \in \mathbb{O}$ is obtained and (3) shows that the statement of the theorem is valid.

(c) Let

$$\left(\frac{\zeta}{\phi(\zeta)}\right)^2 \phi'(\zeta) - 1 = \omega(\zeta). \tag{4}$$

Then, $\omega(\zeta) = d_{n-1}\zeta^{n-1} + \cdots$ is analytic in $\mathbb{O}$. Additionally, from (4) and after logarithmic differentiation, it calculates to

$$1 + \frac{\zeta\phi''(\zeta)}{\phi'(\zeta)} = \frac{\zeta\omega'(\zeta)}{1+\omega(\zeta)} + 2\frac{\zeta\phi'(\zeta)}{\phi(\zeta)} - 1. \tag{5}$$

Relation (5) and the condition (1), lead to

$$\mathrm{Re}\left(\frac{\zeta\omega'(\zeta)}{1+\omega(\zeta)} + 2\frac{\zeta\phi'(\zeta)}{\phi(\zeta)} - 1\right) < 1 + \frac{\alpha}{2}. \tag{6}$$

Additionally, note that if $n \geq 5 + \alpha$, then $n \geq 1 + \alpha$. Hence, (b) shows that

$$\frac{\zeta\phi'(\zeta)}{\phi(\zeta)} \prec \frac{1-\zeta}{1-a\zeta},$$

where $a = 1/(1+\alpha)$, which implies

$$\mathrm{Re}\left(\frac{\zeta\,\phi'(\zeta)}{\phi(\zeta)}\right) > 0.$$

Thus, (6) becomes

$$\mathrm{Re}\left(\frac{\zeta\omega'(\zeta)}{1+\omega(\zeta)} - 1\right) < 1 + \frac{\alpha}{2} - 2\mathrm{Re}\left(\frac{\zeta\,\phi'(\zeta)}{\phi(\zeta)}\right) < 1 + \frac{\alpha}{2}. \tag{7}$$

Now suppose that there exists a point $\zeta_0 \in \mathbb{O}$, such that $|\omega(\zeta_0)| = 1$. Then, by applying Lemma 1, the following is obtained:

$$\zeta_0\omega'(\zeta_0) = k\omega(\zeta_0); \quad (k \geq n-1; \omega(\zeta_0) = e^{i\theta}; \theta \in \mathbb{R}),$$

thus

$$\mathrm{Re}\left(\frac{\zeta_0\omega'(\zeta_0)}{1+\omega(\zeta_0)} - 1\right) = \mathrm{Re}\left(\frac{ke^{i\theta}}{1+e^{i\theta}}\right) - 1 = \frac{k}{2} - 1.$$

Hence,

$$\mathrm{Re}\left(\frac{\zeta_0\omega'(\zeta_0)}{\omega(\zeta_0)} - 1\right) \geq 1 + \frac{\alpha}{2},$$

where $k \geq n-1$, and $n \geq 5 + \alpha$, which is a contradiction to (1). It means that $|\omega(\zeta)| < 1, \zeta \in \mathbb{O}$. Moreover, (4) shows that

$$\left|\left(\frac{\zeta}{\phi(\zeta)}\right)^2 \phi'(\zeta) - 1\right| < 1, \quad \zeta \in \mathbb{O}$$

which concludes the desired $\phi \in \mathcal{U}$.

□

For $\alpha = 1$, we have the next result.

Corollary 1. *If $\phi \in \mathcal{A}_2$ satisfies the condition*

$$\mathrm{Re}\left(1+\frac{\zeta\,\phi''(\zeta)}{\phi'(\zeta)}\right) < \frac{3}{2}, \quad \zeta \in \mathbb{O},$$

then

(a) $\mathrm{Re}\,(\phi'(\zeta)) > 0$, $\zeta \in \mathbb{O}$.

(b) $\frac{\zeta\phi'(\zeta)}{\phi(\zeta)} \prec \frac{1-\zeta}{1-\zeta/2}$.

These are the former results given in [19].

Theorem 2. *Assume that $\alpha \geq 0$ and $\phi \in \mathcal{A}_n, n \geq 2$ satisfy the condition*

$$\mathrm{Re}\left(1+\frac{\zeta\phi''(\zeta)}{\phi'(\zeta)}\right) > -\alpha, \quad \zeta \in \mathbb{O}. \tag{8}$$

(a) *If $n \geq 2\alpha+3$, then $\mathrm{Re}\,(\phi'(\zeta)) > 0$, $\zeta \in \mathbb{O}$.*

(b) *If $n \geq 2\alpha+2$, and $\phi(\zeta)/\zeta \neq 0$, then $\zeta\phi'(\zeta)/\phi(\zeta) \prec 1/(1-\zeta)$*

(c) *For every $n \in \mathbb{N}$, and $n > 2$, there exist $\phi \in \mathcal{A}_n$ and $\phi \in \mathcal{U}$ such that (8) is not satisfied, or equivalently*

$$\mathrm{Re}\left(1+\frac{\zeta\phi''(\zeta)}{\phi'(\zeta)}\right) \to -\infty,$$

where $\zeta \in \mathbb{O}$, ζ is real, and $\zeta \to 1$.

(d) *For $n = 2$, there exists $\phi \in \mathcal{A}_2$ such that (8) is satisfied for every $\alpha > 0$, but $\phi \notin \mathcal{U}$.*

Proof.

(a) Let

$$\phi'(\zeta) = \frac{1}{(1-\omega(\zeta))^{\frac{2(\alpha+1)}{n-1}}}, \tag{9}$$

where $0 < 2(1+\alpha)/(n-1) \leq 1$. Similarly, as in the proof of the previous theorem, $\phi'(\zeta) \neq 0$ for all $\zeta \in \mathbb{O}$, which implies that $\omega(\zeta) = b_{n-1}\zeta^{n-1} + \cdots$ is analytic in $\mathbb{O}$ with $\omega(0) = 0$. Additionally, from (9) and after logarithmic differentiation, it yields

$$\left(1+\frac{\zeta\phi''(\zeta)}{\phi'(\zeta)}\right) = 1 + \frac{2(\alpha+1)(\zeta\omega'(\zeta))}{(n-1)(1-\omega(\zeta))}. \tag{10}$$

Suppose that there exists a point $\zeta_0 \in \mathbb{O}$, such that

$$\max_{|\zeta|\leq|\zeta_0|} |\omega(\zeta)| = |\omega(\zeta_0)| = 1.$$

Then, by applying Lemma 1, the following is acquired:

$$\zeta_0\omega'(\zeta_0) = k\omega(\zeta_0); \quad (k \geq n-1; \omega(\zeta_0) = e^{i\theta}; \theta \in \mathbb{R}).$$

Thus,

$$\begin{aligned}\mathrm{Re}\left(1+\frac{\zeta_0\phi''(\zeta_0)}{\phi'(\zeta_0)}\right) &= 1 + \frac{2(\alpha+1)}{n-1}\,\mathrm{Re}\left(\frac{ke^{i\theta}}{1-e^{i\theta}}\right)\\ &= 1 - \frac{k(\alpha+1)}{n-1}\\ &\leq -\alpha,\end{aligned}$$

which is a contradiction to (8). That implies $|\omega(\zeta)| < 1, \zeta \in \mathbb{O}$. Furthermore, (9) shows that

$$\begin{aligned}|\arg(\phi'(\zeta))| &= \frac{2(\alpha+1)}{n-1}|\arg((1-\omega(\zeta))| \\ &< \frac{2(\alpha+1)}{n-1}\frac{\pi}{2} \\ &\leq \frac{\pi}{2},\end{aligned}$$

where $n \geq 2\alpha + 3$. Hence, $\operatorname{Re}(\phi'(\zeta)) > 0$, $\zeta \in \mathbb{O}$.

(b) Define a function ω by

$$\frac{\zeta\phi'(\zeta)}{\phi(\zeta)} = \frac{1}{1-\omega(\zeta)}. \tag{11}$$

Then, $\omega(\zeta) = c_{n-1}\zeta^{n-1} + \ldots$ is analytic in $\mathbb{O}$. Additionally, suppose that there exists a point $\zeta_0 \in \mathbb{O}$, such that

$$\max_{|\zeta|\leq|\zeta_0|} |\omega(\zeta)| = |\omega(\zeta_0)| = 1.$$

By applying Lemma 1,

$$\zeta_0\omega'(\zeta_0) = k\omega(\zeta_0); \quad (k \geq n-1; \omega(\zeta_0) = e^{i\theta}; \theta \in \mathbb{R}).$$

Therefore, (11) leads to

$$\begin{aligned}\operatorname{Re}\left(1 + \frac{\zeta_0\phi''(\zeta_0)}{\phi'(\zeta_0)}\right) &= \operatorname{Re}\left(\frac{\zeta_0\omega'(\zeta_0)}{1-\omega(\zeta_0)} + \frac{1}{1-\omega(\zeta_0)}\right) \\ &= \operatorname{Re}\left(\frac{ke^{i\theta}+1}{1-e^{i\theta}}\right) = \frac{1-k}{2} \leq -\alpha,\end{aligned}$$

where

$$k - 1 \geq (n-1) - 1 = n - 2 \geq (2\alpha+2) - 2 = 2\alpha.$$

Which is a contradiction to (8). Thus, $|\omega(\zeta)| < 1, \zeta \in \mathbb{O}$ and (11) shows that the statement of the theorem is valid.

(c) For $n > 2$, let $\phi \in \mathcal{A}_n$ be defined by

$$\frac{\zeta}{\phi(\zeta)} = 1 + \frac{1}{n-2}\zeta^{n-1}.$$

Since

$$\left|\left(\frac{\zeta}{\phi(\zeta)}\right)^2\phi'(\zeta) - 1\right| = |\zeta^{n-1}| < 1,$$

then $\phi \in \mathcal{U}$. After logarithmic differentiation, the following is obtained:

$$1 + \frac{\zeta\,\phi''(\zeta)}{\phi'(\zeta)} = \left(1 - \frac{(n-1)\zeta^{n-2}}{1-\zeta^{n-2}} - 2\frac{(n-1)\zeta^{n-1}}{(n-2)+\zeta^{n-1}}\right). \tag{12}$$

Thus, when ζ is real, $\zeta \to 1$ and $\zeta \in \mathbb{O}$, we have

$$\operatorname{Re}\left(1 + \frac{\zeta\phi''(\zeta)}{\phi'(\zeta)}\right) \to -\infty.$$

(d) Let $\phi \in \mathcal{A}_2$ be defined by $\phi(\zeta) = -\ln(1-\zeta)$. Then,

$$1 + \frac{\zeta\phi''(\zeta)}{\phi'(\zeta)} = \frac{1}{1-\zeta},$$

which implies that

$$\mathrm{Re}\left(1 + \frac{\zeta\phi''(\zeta)}{\phi'(\zeta)}\right) > \frac{1}{2} > -\alpha.$$

Since

$$\left|\left(\frac{\zeta}{\phi(\zeta)}\right)^2 \phi'(\zeta) - 1\right| = \left|\left(\frac{-\zeta}{\ln(1-\zeta)}\right)^2 \frac{1}{1-\zeta} - 1\right|,$$

and $(1-\zeta)\ln^2(1-\zeta) \to 0$ when $\zeta \in \mathbb{O}$, $\zeta \to 1$ is real, it follows that

$$\left|\left(\frac{\zeta}{\phi(\zeta)}\right)^2 \phi'(\zeta) - 1\right| > 1, \quad \zeta \in \mathbb{O}.$$

Hence, $\phi \notin \mathcal{U}$.

□

For $\alpha = 0$ and $\alpha = 1/2$, Theorem 2 leads to the next corollaries.

Corollary 2. *If $\phi \in \mathcal{A}_n$ satisfies the condition*

$$\mathrm{Re}\left(1 + \frac{\zeta\phi''(\zeta)}{\phi'(\zeta)}\right) > 0, \quad \zeta \in \mathbb{O},$$

(a) Then, $\mathrm{Re}\,(\phi'(\zeta)) > 0$, $\zeta \in \mathbb{O}$, *whenever* $n \geq 3$.
(b) Then, $\mathrm{Re}\,(\zeta\phi'(\zeta)/\phi(\zeta)) > 1/2, \zeta \in \mathbb{O}$, *whenever* $n \geq 2$ *and* $\phi(\zeta)/\zeta \neq 0$.

Remark 1. *For the function $\phi(\zeta) = \zeta/(1-\zeta) \in \mathcal{A}_2$, we have $\phi'(\zeta) = 1/(1-\zeta)^2$ and it has a non-positive real part in $\mathbb{O}$, which means that the result (a) is the best possible. The result (b) is well-known.*

Corollary 3. *If $\phi \in\in \mathcal{A}_n$ satisfies the condition*

$$\mathrm{Re}\left(1 + \frac{\zeta\,\phi''(\zeta)}{\phi'(\zeta)}\right) > -\frac{1}{2}, \quad \zeta \in \mathbb{O}.$$

(a) Then, $\mathrm{Re}\,(\phi'(\zeta)) > 0$, $\zeta \in \mathbb{O}$, *whenever* $n \geq 4$.
(b) Then, $\mathrm{Re}\,(\zeta\phi'(\zeta)/\phi(\zeta)) > 1/2, \zeta \in \mathbb{O}$, *whenever* $n \geq 3$ *and* $\phi(\zeta)/\zeta \neq 0$.

Remark 2. *The result (b) for $n = 3$ is given in ([20] (Theorem 2.6i, p. 68)).*

Theorem 3. *If $\alpha > 0$ and $\phi \in \mathcal{A}_n$ satisfy the condition*

$$|\phi''(\zeta)| \leq \alpha, \quad \zeta \in \mathbb{O}, \tag{13}$$

then for $n \geq \alpha + 1$, $\mathrm{Re}\,\phi'(\zeta) > 0, \zeta \in \mathbb{O}$, that is, $|\phi'(\zeta) - 1| < 1$.

Proof. From (13) we have that

$$|\zeta\phi''(\zeta)| \leq \alpha|\zeta| < \alpha, \zeta \in \mathbb{O}. \tag{14}$$

Putting

$$\phi'(\zeta) = 1 - w(\zeta). \tag{15}$$

Then, w is analytic in $\mathbb{O}$ with $w(0) = 0$, and we want to prove that $|w(\zeta)| < 1, \zeta \in \mathbb{O}$. If not, then by Lemma 1, there exists a point $\zeta_0 \in \mathbb{O}$ such that

$$\zeta_0 w'(\zeta_0) = k w(\zeta_0), k \geq 1, w(\zeta_0) = e^{i\theta}, \theta \in \mathbb{R}.$$

Now, by (15) we have

$$\left|\zeta_0 \phi''(\zeta_0)\right| = \left|-\zeta_0 w'(\zeta_0)\right| = k|w(\zeta_0)| \geq n - 1 \geq \alpha,$$

which is a contradiction to (13). It means that $|w(\zeta)| < 1$ and from (15) that $|\phi'(\zeta) - 1| < 1$, $\zeta \in \mathbb{O}$. □

Remark 3. *Since*

$$\phi'(\zeta) - 1 = \int_0^\zeta \phi''(\zeta) d\zeta$$

and $|\phi''(\zeta)| < \alpha$, *then*

$$|\phi'(\zeta) - 1| = |\int_0^\zeta \phi''(\zeta) d\zeta| < \alpha|\zeta| < 1$$

only if $0 < \alpha \leq 1$. *However, from Theorem 3 it is true for all* $\alpha > 0$ *with the condition* $n \geq 1 + \alpha$. *For example, if* $\alpha = 2$, *i.e.,* $|\phi''(\zeta)| < 2, \zeta \in \mathbb{O}$, *then for* $n \geq 3$, *we have* $\phi(\zeta) = \zeta + \phi_3 \zeta^3 + \cdots$.

4. Conclusions

From above, we proposed a new subclass of analytic normalized functions in the open unit disk. We presented a collection of results discussing the univalency and the stralikeness of the new class. Moreover, some recent works are indicated under our main results as consequences. For future work, one can make a development for the suggested class in view of other classes of analytic functions, such as the meromorphic, p-valent and harmonic classes. As an application, one can consider the suggested class operating with the class of special functions.

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