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A NOTE ON r–CIRCULANT MATRICES INVOLVING GENERALIZED NARAYANA NUMBERS

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Abstract. In order to further connect structured matrices and integer sequences, r-circulant matrices involving the generalized Narayana numbers are considered. Estimates for spectral norms bounds of such matrices are presented and their eigenvalues are determined. Moreover, the conditions under which the circulant matrix and the skew circulant matrix involving generalized Narayana numbers are invertible are given. In particular, it is shown that every circulant matrix with Narayana numbers is necessarily invertible.

1. Introduction

In the middle of the 14^{th} century, the Indian mathematician Narayana Pandit posed the following problem: *A cow produces one calf every year. Beginning in its fourth year, each calf produces one calf at the beginning of each year. How many cows and calves are there altogether after 20 years?*

This problem leads to the *Narayana cows sequence*, also known as the *supergolden sequence*, which is given by a third-order recurrence relation:

$$\mathcal{N}_n = \mathcal{N}_{n-1} + \mathcal{N}_{n-3}, \quad n \geqslant 3, \tag{1}$$

where $\mathcal{N}_0 = 0$, $\mathcal{N}_1 = 1$ and $\mathcal{N}_2 = 1$. Thus, the initial values of the Narayana cows sequence are:

$$0,1,1,1,2,3,4,6,9,13,19,28,41,60,88,129,\ldots \quad \text{(sequence A000930 in [22])}$$

The Narayana cows sequence, or Narayana sequence for short, is probably less well known than the famous Fibonacci sequence given by $f_n = f_{n-1} + f_{n-2}$, $n \geqslant 2$, with initial values $f_0 = 0$, $f_1 = 1$, but these sequences are closely related. That is why it is often called the Fibonacci–Narayana sequence. It is well known that the Fibonacci sequence has a wide application in various fields, from biology to art and architecture, but how about the Narayana cows sequence?

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In recent years, starting with [1], the Narayana sequence has been the subject of great interest. It turns out that Narayana's numbers also has a wide range of important applications especially in automata theory, as well as the availability for additional research. We shall try to give as brief a summary of them.

Some basic properties of the Fibonacci–Narayana numbers and the generalized Fibonacci–Narayana quaternions are presented in [12]. There is a rich literature concerning different types of generalizations of the Narayana sequence. As a generalization, Ramirez and Sirvent in [21] introduced the k-Narayana numbers and studied their properties via matrix methods. Another generalization, called the generalized order k-Narayana's cows sequence, was given by Bilgici in [3]. T. Goy in [13] established connection between the Fibonacci–Narayana numbers and the Fibonacci numbers via the Toeplitz–Hessenberg determinants. Some relations between the generalized Fibonacci–Narayana sequences and one type of upper Hessenberg matrix are studied in [20]. The Narayana's sequence also has interesting properties which can be used in cryptographic and key generation applications. Kirthi and Kak (see [17]) presented one method of universal coding based on the Narayana's sequence. An interesting application of the Narayana's sequence to the construction of optimal gate circuits that can be used for quantum computation is given in [28].

On the other hand, structured matrices whose entries are some of the well-known integer sequences have been extensively studied in recent years. A family of structured matrices such as Toeplitz, Hankel, Vandermonde, circulant, Hessenberg, Cauchy and other well-known families of special matrices naturally arise in different areas, such as data approximation, physics, engineering, economics, signal and image processing, numerical analysis, communications, error correcting code theory, statistics, and so forth.

Given the importance of the Narayana's cows sequence, as well as the importance of structured matrices, it seems interesting to consider r-circulant matrices involving the Narayana numbers.

The plan of the paper is the following: In Section 2, we define the generalized Narayana sequence and establish its basic properties. In particular, the identities established by Lemma 1 and Proposition 2 are necessary for the rest of the paper. In this section we define r-circulant matrices and give a brief overview of their importance in applications as well as previous results on this topic.

Section 3 contains the main results: Theorem 1 and Theorem 2, as well as Theorem 3 and Theorem 4 which can be considered as the corollaries of Theorem 2. However, these results are important itself.

2. Preliminaries

A special type of Toeplitz matrix, such that each row is a circular shift of the first row, is called a *circulant matrix*. In recent years, a particular attention has been paid on circulant matrices and their relatives. For some of the results concerning this class of matrices the reader may wish to consult [9, 10, 14]. According to Davis ([10]), the r-circulant matrix is defined as follows.

DEFINITION 1. Let $n \geqslant 2$ be an integer, $u = (c_0, c_1, \ldots, c_{n-1})^T \in \mathbb{C}^n$, and $r \in \mathbb{C}$. A matrix $C = \mathrm{Circ}_r(u) \in \mathbb{M}_n(\mathbb{C})$ is called a *r-circulant matrix* if

$$c_{ij} = \begin{cases} c_{j-i}, & j \geqslant i, \\ rc_{n+j-i}, & j < i, \end{cases}$$

i.e., if C has the following form:

$$C = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ rc_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ rc_{n-2} & rc_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ rc_2 & rc_3 & rc_4 & \cdots & c_0 & c_1 \\ rc_1 & rc_2 & rc_3 & \cdots & rc_{n-1} & c_0 \end{bmatrix}. \tag{2}$$

Thus, an r-circulant matrix is fully specified by one vector $(c_0, c_1, \ldots, c_{n-1})^T$ and the parameter r. We will write $C = \mathrm{Circ}_r(c_0, c_1, c_2, \ldots, c_{n-1})$, or C_r for short, if the order of the matrix is known. It turned out that r-circulant matrices have many useful properties, and have become one of the most important research subjects in the fields of pure and applied mathematics.

Hence, the circulant matrices are r-circulant matrices if $r = 1$ and *skew circulant matrices* are r-circulant matrices if $r = -1$. In particular, 0-circulant matrix is an upper triangular Toeplitz matrix (which is sometimes called a *semicirculant* matrix). In what follows, we assume that r is nonzero.

Recently, there have been many papers on r-circulant matrices involving integer sequences. For example, Solak in [25] have found the bounds of spectral norms of circulant matrices with the Fibonacci and Lucas numbers. In [23] Shen and Cen studied the bounds for the spectral norms of r-circulant matrices involving the Fibonacci and Lucas numbers. Some improvements of previous results on r-circulant matrices involving the Fibonacci numbers are presented in [19]. The paper [6] deals with circulant matrices with the Jacobsthal and Jacobsthal–Lucas numbers. Bozkurt and Tam obtained some useful formulas for the determinants and inverses of r-circulant matrices involving an arbitrary second order recurrence sequence (see [7]). In [4] E. Boman derived a simple formula for the Moore–Penrose pseudoinverse of a general $n \times n$ r-circulant matrix. In [24] and [29] the authors considered the norms of r-circulant and geometric circulant matrices with the generalized r-Horadam numbers. For some other results on r-circulants whose entries are various integer sequences we recommend [2, 16, 26].

In this paper, we explore the r-circulant matrices

$$\mathrm{Circ}_r(\mathscr{N}_0, \mathscr{N}_1, \mathscr{N}_2, \ldots, \mathscr{N}_{n-1}) \tag{3}$$

where $n \geqslant 2$ is a positive integer and $\mathscr{N}_i$ is the i-th Narayana number.

We estimate the upper and lower bounds for the norms of these matrices and designate their eigenvalues. In addition, we obtain similar results for r-circulant matrices

$$\mathrm{Circ}_r(\mathrm{u}_0, \mathrm{u}_1, \mathrm{u}_2, \ldots, \mathrm{u}_{n-1}) \tag{4}$$

involving generalized Narayana's type numbers defined by (7).

Moreover, if $n \geqslant 2$, as a consequence of the obtained results we get that all circulant matrices involving Narayana numbers are invertible, while skew circulant matrices $\mathrm{Circ}_{-1}(\mathscr{N}_0, \mathscr{N}_1, \ldots, \mathscr{N}_{n-1})$ are invertible if and only if $n \neq 3$.

The Binet formula allows the expression of a sequence $\{\mathscr{N}_n\}_{n\in\mathbb{N}}$ as a function of the roots α, β and γ of the characteristic equation

$$x^3 - x^2 - 1 = 0. \tag{5}$$

If $\Delta = \sqrt[3]{\frac{2}{29+3\sqrt{93}}}$ and $\omega_3 = \frac{1+i\sqrt{3}}{2}$, then the roots of the characteristic equation (5) are $\alpha = \frac{1}{3}(1+\Delta+\Delta^{-1})$, $\beta = \frac{1}{3}(1-\omega_3\Delta+\omega_3^2\Delta^{-1})$ and $\gamma = \frac{1}{3}(1+\omega_3^2\Delta-\omega_3\Delta^{-1})$. Thus, the n–th Narayana number is (see [12, Theorem 3.3])

$$\mathscr{N}_n = \frac{\alpha^{n+1}(\gamma-\beta)+\beta^{n+1}(\alpha-\gamma)+\gamma^{n+1}(\beta-\alpha)}{(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)}. \tag{6}$$

One can generalize the Narayana's sequence by varying the initial conditions. Let us define the *generalized Narayana's type of sequence* $\{\mathsf{u}_n\}_{n\in\mathbb{N}}$ as follows:

$$\mathsf{u}_n = \mathsf{u}_{n-1} + \mathsf{u}_{n-3}, \quad n \geqslant 3, \tag{7}$$

where $\mathsf{u}_0 = a$, $\mathsf{u}_1 = b$, $\mathsf{u}_2 = c$, for some nonnegative integers a, b and c such that $a^2+b^2+c^2 \neq 0$. Clearly,

$$\mathsf{u}_n = a\mathscr{N}_{n-2} + b\mathscr{N}_{n-3} + c\mathscr{N}_{n-1}, \; n \geqslant 3. \tag{8}$$

Namely, $\mathsf{u}_3 = a+c = a\mathscr{N}_1 + b\mathscr{N}_0 + c\mathscr{N}_2$. From the preceding, the required result can be easily proved by induction.

The previous equation establishes one useful connection between the generalized Narayana numbers and the ordinary Narayana numbers. Essentially, the properties of the generalized Narayana numbers are determined by the properties of the Narayana sequence $\{\mathscr{N}_n\}_{n\in\mathbb{N}}$.

For the sake of completeness, let us mention another way to obtain equality (8). Since $\{\mathsf{u}_n\}_{n\in\mathbb{N}}$ satisfy a recurrence relation (7), it follows that

$$\begin{bmatrix} \mathsf{u}_n \\ \mathsf{u}_{n+1} \\ \mathsf{u}_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}^n \begin{bmatrix} \mathsf{u}_0 \\ \mathsf{u}_1 \\ \mathsf{u}_2 \end{bmatrix} = \begin{bmatrix} \mathscr{N}_{n-2} & \mathscr{N}_{n-3} & \mathscr{N}_{n-1} \\ \mathscr{N}_{n-1} & \mathscr{N}_{n-2} & \mathscr{N}_n \\ \mathscr{N}_n & \mathscr{N}_{n-1} & \mathscr{N}_{n+1} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \; n \geqslant 3,$$

whence $\mathsf{u}_n = a\mathscr{N}_{n-2} + b\mathscr{N}_{n-3} + c\mathscr{N}_{n-1}$, for $n \geqslant 3$, as requested.

The specific values of a, b and c give various Narayana's type integer sequences. Some examples of the first few values of the generalized Narayana's type sequences listed in The On-Line Encyclopedia of Integer Sequences (OEIS) are given in Table 1.

	a	b	c	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}	u_{11}	u_{12}	OEIS
$\{\mathscr{N}_n\}_{n\in\mathbb{N}}$	0	1	1	1	2	3	4	6	9	13	19	28	41	A000930
$\{\mathscr{A}_n\}_{n\in\mathbb{N}}$	3	1	1	4	5	6	10	15	21	31	46	67	98	A001609
$\{\mathscr{B}_n\}_{n\in\mathbb{N}}$	1	2	2	3	5	7	10	15	22	32	47	69	101	A097333
$\{\mathscr{C}_n\}_{n\in\mathbb{N}}$	2	1	2	4	5	7	11	16	23	34	50	73	107	A164316

Table 1: *The first few terms of the generalized Narayana-type sequences.*

Since the characteristic equation (5) remains unchanged, the Binet formula for the generalized Narayana's sequence becomes

$$\mathsf{u}_n = X\alpha^n + Y\beta^n + Z\gamma^n,\ n \geqslant 3, \tag{9}$$

where $\mathsf{u}_0 = a$, $\mathsf{u}_1 = b$, $\mathsf{u}_2 = c$, and

$$X = \frac{a\beta\gamma - b(\beta+\gamma)+c}{(\alpha-\gamma)(\alpha-\beta)},\quad Y = -\frac{a\alpha\gamma - b(\alpha+\gamma)+c}{(\beta-\gamma)(\alpha-\beta)},\quad Z = \frac{a\alpha\beta - b(\alpha+\beta)+c}{(\alpha-\gamma)(\beta-\gamma)}.$$

First, we are going to prove some results on the properties of the generalized Narayana numbers necessary for the rest of the paper. Note that some parts of these results are similar to the results for the ordinary Narayana numbers obtained in [11].

LEMMA 1. *Let $\{\mathscr{N}_n\}_{n\in\mathbb{N}}$ and $\{\mathsf{u}_n\}_{n\in\mathbb{N}}$ be the Narayana cows sequence and the generalized Narayana sequence. Then, each of the following identities holds true:*

1°) $S_n := \mathsf{u}_0 + \mathsf{u}_1 + \mathsf{u}_2 + \cdots + \mathsf{u}_n = \mathsf{u}_{n+3} - c$.

2°) $M_{2n} := \mathsf{u}_0 + \mathsf{u}_2 + \mathsf{u}_4 + \cdots + \mathsf{u}_{2n} = \frac{1}{3}(\mathsf{u}_{2n+4} + \mathsf{u}_{2n} + a - b - c)$.

3°) $M_{2n+1} := \mathsf{u}_1 + \mathsf{u}_3 + \mathsf{u}_5 + \cdots + \mathsf{u}_{2n+1} = \frac{1}{3}(2\mathsf{u}_{2n+4} - \mathsf{u}_{2n} - a + b - 2c)$.

4°) $\mathsf{u}_{n+m} = \mathsf{u}_{n-1}\mathscr{N}_{m+2} + \mathsf{u}_{n-2}\mathscr{N}_m + \mathsf{u}_{n-3}\mathscr{N}_{m+1}$, $n \geqslant 3$, $m \geqslant 0$.

Proof. The first identity follows immediately since $\sum_{k=0}^{n}(\mathsf{u}_{k+3} - \mathsf{u}_{k+2} - \mathsf{u}_k) = 0$. Therefore,

$$M_{2n+1} + M_{2n} = S_{2n+1} = u_{2n+4} - c. \tag{10}$$

On the other hand, $\sum_{k=0}^{n}(\mathsf{u}_{2k+3} - \mathsf{u}_{2k+2} - \mathsf{u}_{2k}) = 0$. Hence, it follows that

$$M_{2n+1} - 2M_{2n} = \mathsf{u}_{2n+2} - \mathsf{u}_{2n+3} + b - a. \tag{11}$$

Thus, the solution of a system of linear equations (10), (11) yields identities 2°) and 3°).

If $m = 0$, then equation 4°) is reduced to the identity $\mathsf{u}_n = \mathsf{u}_{n-1} + \mathsf{u}_{n-3}$. The rest of the proof follows easily by induction using (8). □

COROLLARY 1. (See [11]) *The Narayana numbers satisfies the following identity:*

$$\mathcal{N}_{2n} = \mathcal{N}_{n+1}^2 + \mathcal{N}_{n-1}^2 - \mathcal{N}_{n-2}^2. \tag{12}$$

Proof. Assume that $n = k+1$ and $m = k-1$. According to the previous lemma (part 4°)), we have

$$\begin{aligned}\mathcal{N}_{2k} &= \mathcal{N}_k\mathcal{N}_{k+1} + \mathcal{N}_{k-1}\mathcal{N}_{k-1} + \mathcal{N}_{k-2}\mathcal{N}_k = \mathcal{N}_{k-1}^2 + \mathcal{N}_k(\mathcal{N}_{k+1} + \mathcal{N}_{k-2})\\ &= \mathcal{N}_{k-1}^2 + (\mathcal{N}_{k+1} - \mathcal{N}_{k-2})(\mathcal{N}_{k+1} + \mathcal{N}_{k-2}) = \mathcal{N}_{k-1}^2 + \mathcal{N}_{k+1}^2 - \mathcal{N}_{k-2}^2,\end{aligned}$$

as required. □

PROPOSITION 1. *Let $n \geqslant 2$ be an integer. The sum of the squares of the first $n+1$ Narayana numbers is:*

$$D_n = \mathcal{N}_0^2 + \mathcal{N}_1^2 + \cdots + \mathcal{N}_n^2 = \frac{1}{3}(\mathcal{N}_{2n+2} + \mathcal{N}_{2n-2} + 1) - \mathcal{N}_{n-2}^2. \tag{13}$$

Proof. By property (12), we have

$$\sum_{k=2}^{n-1} \mathcal{N}_{2k} = \sum_{k=2}^{n-1} \mathcal{N}_{k-1}^2 + \sum_{k=2}^{n-1} \mathcal{N}_{k+1}^2 - \sum_{k=2}^{n-1} \mathcal{N}_{k-2}^2.$$

Therefore, we get $M_{2n-2} - \mathcal{N}_2 - \mathcal{N}_0 = D_n - 2\mathcal{N}_0^2 - \mathcal{N}_1^2 - \mathcal{N}_2^2 + \mathcal{N}_{n-2}^2$. According to the initial conditions, $\mathcal{N}_0 = 0$, $\mathcal{N}_1 = \mathcal{N}_2 = 1$, hence

$$D_n = M_{2n-2} + 1 - \mathcal{N}_{n-2}^2.$$

Applying Lemma 1 to the previous equation, we conclude the proof. □

According to (13), it is easy to obtain the sum $\sum_{k=0}^{n} \mathcal{N}_k^2$. However, given the change in initial values, there is no such nice formula for $\sum_{k=0}^{n} \mathsf{u}_k^2$.

PROPOSITION 2. *Let $n \geqslant 1$ be an integer. If we define $P_n := \sum\limits_{k=0}^{n} \mathsf{u}_k^2$, then*

$$P_n = \frac{1}{3}((2a+2b-c)^2 - 2a(5b-c) + 2\mathsf{u}_n\mathsf{u}_{n+1} + 4\mathsf{u}_{n-1}\mathsf{u}_{n+1} + 4\mathsf{u}_n\mathsf{u}_{n+2} - \mathsf{u}_{n+3}^2). \tag{14}$$

Proof. Let $n \geqslant 1$ be an integer, $P_n = \sum_{k=0}^{n} \mathsf{u}_k^2$, $Q_n = \sum_{k=0}^{n} \mathsf{u}_k\mathsf{u}_{k+2}$ and $R_n = \sum_{k=0}^{n} \mathsf{u}_k\mathsf{u}_{k+1}$. According to (7), we have $\mathsf{u}_k = \mathsf{u}_{k+3} - \mathsf{u}_{k+2}$ and consequently:

$$\begin{aligned}P_n &= \sum_{k=0}^{n} (\mathsf{u}_{k+3} - \mathsf{u}_{k+2})^2 = \sum_{k=0}^{n} \mathsf{u}_{k+3}^2 + \sum_{k=0}^{n} \mathsf{u}_{k+2}^2 - 2\sum_{k=0}^{n} \mathsf{u}_{k+2}(\mathsf{u}_{k+2} + \mathsf{u}_k)\\ &= -c^2 + \mathsf{u}_{n+3}^2 - 2Q_n,\\ Q_n &= \sum_{k=0}^{n} \mathsf{u}_k\mathsf{u}_{k+2} = ac + \sum_{k=1}^{n} \mathsf{u}_k(\mathsf{u}_{k+1} + \mathsf{u}_{k-1}) = ac - ab + 2R_n - \mathsf{u}_n\mathsf{u}_{n+1}.\end{aligned}$$

Similarly, as $\mathsf{u}_k = \mathsf{u}_{k+1} - \mathsf{u}_{k-2}$, and therefore $\mathsf{u}_k^2 = \mathsf{u}_k\mathsf{u}_{k+1} - \mathsf{u}_k\mathsf{u}_{k-2}$, we obtain

$$\begin{aligned} P_n - a^2 - b^2 &= \sum_{k=2}^{n} \mathsf{u}_k\mathsf{u}_{k+1} - \sum_{k=2}^{n} \mathsf{u}_k\mathsf{u}_{k-2} \\ &= -ab - bc + R_n - Q_n + \mathsf{u}_{n-1}\mathsf{u}_{n+1} + \mathsf{u}_n\mathsf{u}_{n+2}. \end{aligned}$$

The previous identities implies the system of linear equations

$$\begin{array}{rcl} P_n + 2Q_n & = & \mathsf{u}_{n+3}^2 - c^2 \\ Q_n - 2R_n & = & a(c-b) - \mathsf{u}_n\mathsf{u}_{n+1} \\ P_n + Q_n - R_n & = & a^2 + b^2 - (a+c)b + \mathsf{u}_{n-1}\mathsf{u}_{n+1} + \mathsf{u}_n\mathsf{u}_{n+2}, \end{array}$$

whence we get the required formula (14), as well as the summation formulas for Q_n and R_n. □

REMARK 1. Note that formula (14) can be expressed in terms of the initial conditions a, b, c and the ordinary Narayana's sequence $\{\mathscr{N}_n\}_{n\in\mathbb{N}}$ using the identity (8). It is also obvious that (13) follows directly from (14) for $a=0$, $b=1$ and $c=1$. However, we still decided to include the proof of Proposition 1, which better reflects some nice properties of Narayana's numbers.

DEFINITION 2. Let $n \geqslant 2$ be an integer, $\mathsf{u} = (\mathsf{u}_0, \mathsf{u}_1, \ldots, \mathsf{u}_{n-1})^T$, and $C = \mathrm{Circ}(\mathsf{u})$. The *representer* $\Phi_{n,\mathsf{u}}$ of the circulant matrix C is a polynomial in the indeterminate t defined by:

$$\Phi_{n,\mathsf{u}}(t) := \sum_{k=0}^{n-1} \mathsf{u}_k t^k.$$

The following proposition gives an explicit formula for the representer $\Phi_{n,\mathsf{u}}$.

PROPOSITION 3. *Let $n \geqslant 3$ be an integer and $\mathsf{u} = (\mathsf{u}_0, \mathsf{u}_1, \ldots, \mathsf{u}_{n-1})^T$. Then*

$$(1 - t - t^3)\Phi_{n,\mathsf{u}}(t) = a + (b-a)t + (c-b)t^2 - \mathsf{u}_n t^n - \mathsf{u}_{n-2}t^{n+1} - \mathsf{u}_{n-1}t^{n+2}.$$

Proof. Note that $\mathsf{u}_k = \mathsf{u}_{k-1} + \mathsf{u}_{k-3}$, for all $k \geqslant 3$. Therefore, we obtain

$$\begin{aligned} \Phi_{n,\mathsf{u}}(t) &= \mathsf{u}_0 + \mathsf{u}_1 t + \mathsf{u}_2 t^2 + \sum_{k=3}^{n-1} (\mathsf{u}_{k-1} + \mathsf{u}_{k-3}) t^k \\ &= a + bt + ct^2 + t \sum_{k=2}^{n-2} \mathsf{u}_k t^k + t^3 \sum_{k=0}^{n-4} \mathsf{u}_k t^k \\ &= a + bt + ct^2 + t(\Phi_{n,\mathsf{u}}(t) - a - bt - \mathsf{u}_{n-1}t^{n-1}) + t^3(\Phi_{n,\mathsf{u}}(t) - \sum_{k=1}^{3} \mathsf{u}_{n-k}t^{n-k}) \\ &= (t+t^3)\Phi_{n,\mathsf{u}}(t) + a + (b-a)t + (c-b)t^2 - \mathsf{u}_n t^n - \mathsf{u}_{n-1}t^{n+2} - \mathsf{u}_{n-2}t^{n+1}, \end{aligned}$$

as required. □

COROLLARY 2. *Let $n \geqslant 3$ be an integer and $\mathscr{N} = (\mathscr{N}_0, \mathscr{N}_1, \mathscr{N}_2, \ldots, \mathscr{N}_{n-1})^T$. Then*

$$(1-t-t^3)\Phi_{n,\mathscr{N}}(t) = t - \mathscr{N}_n t^n - \mathscr{N}_{n-2} t^{n+1} - \mathscr{N}_{n-1} t^{n+2}.$$

According to (8), every vector $\mathsf{u} = (\mathsf{u}_0, \mathsf{u}_1, \mathsf{u}_2, \ldots, \mathsf{u}_{n-1})^T$ depends only on the initial values $a = \mathsf{u}_0$, $b = \mathsf{u}_1$, $c = \mathsf{u}_2$, and the properties of the Narayana sequence $\{\mathscr{N}_n\}_{n\in\mathbb{N}}$. Therefore, it can be expressed as a linear combination of vectors:

$$\begin{aligned}
\mathsf{A} &:= (1, 0, \mathscr{N}_0, \mathscr{N}_1, \mathscr{N}_2, \mathscr{N}_3, \ldots, \mathscr{N}_{n-3})^T, \\
\mathsf{B} &:= (0, 1, 0, \mathscr{N}_0, \mathscr{N}_1, \mathscr{N}_2, \ldots, \mathscr{N}_{n-4})^T, \\
\mathsf{C} &:= (0, \mathscr{N}_0, \mathscr{N}_1, \mathscr{N}_2, \mathscr{N}_3, \mathscr{N}_4, \ldots, \mathscr{N}_{n-2})^T,
\end{aligned}$$

i.e., $\mathsf{u} = a\mathsf{A} + b\mathsf{B} + c\mathsf{C}$. In particular, $\mathscr{N} = \mathsf{B} + \mathsf{C}$. Hence,

$$\Phi_{n,\mathsf{u}}(t) = a\Phi_{n,\mathsf{A}}(t) + b\Phi_{n,\mathsf{B}}(t) + c\Phi_{n,\mathsf{C}}(t),$$

where $\Phi_{n,\mathsf{A}}$, $\Phi_{n,\mathsf{B}}$ and $\Phi_{n,\mathsf{C}}$ are determined by

$$\begin{aligned}
\Phi_{n,\mathsf{A}}(t) &= t^2\Phi_{n,\mathscr{N}}(t) + 1 - \mathscr{N}_{n-2}t^n - \mathscr{N}_{n-1}t^{n+1}, \\
\Phi_{n,\mathsf{B}}(t) &= t^3\Phi_{n,\mathscr{N}}(t) + t - \mathscr{N}_{n-3}t^n - \mathscr{N}_{n-2}t^{n+1} - \mathscr{N}_{n-1}t^{n+2}, \\
\Phi_{n,\mathsf{C}}(t) &= t\ \Phi_{n,\mathscr{N}}(t) - \mathscr{N}_{n-1}t^n.
\end{aligned}$$

If r is a nonzero complex number, one can observe a matrix polynomial $\sum_{k=0}^{n-1} \mathsf{u}_k T^k$, where T is $n \times n$ matrix defined by

$$T := \begin{bmatrix}
0 & 1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 1 \\
r & 0 & 0 & 0 & \cdots & 0 & 0
\end{bmatrix}.$$

Obviously,

$$\mathrm{Circ}_r(\mathsf{u}_0, \mathsf{u}_1, \mathsf{u}_2, \ldots, \mathsf{u}_{n-1}) = \Phi_{n,\mathsf{u}}(T) = a\Phi_{n,\mathsf{A}}(T) + b\Phi_{n,\mathsf{B}}(T) + c\Phi_{n,\mathsf{C}}(T).$$

Since $T^n = rI_n$, where I_n is the corresponding identity matrix, it is easily verified that the related matrices $\Phi_{n,\mathsf{A}}(T)$, $\Phi_{n,\mathsf{B}}(T)$ and $\Phi_{n,\mathsf{C}}(T)$ are determined by:

$$\begin{aligned}
\Phi_{n,\mathsf{A}}(T) &= T^2\Phi_{n,\mathscr{N}}(T) + (1 - r\mathscr{N}_{n-2})I_n - r\mathscr{N}_{n-1}T, \\
\Phi_{n,\mathsf{B}}(T) &= T^3\Phi_{n,\mathscr{N}}(T) - r\mathscr{N}_{n-3}I_n + (1 - r\mathscr{N}_{n-2})T - r\mathscr{N}_{n-1}T^2, \\
\Phi_{n,\mathsf{C}}(T) &= T\Phi_{n,\mathscr{N}}(T) - r\mathscr{N}_{n-1}I_n.
\end{aligned}$$

Also note that $\Phi_{n,\mathscr{N}}(T) = \Phi_{n,\mathsf{B}}(T) + \Phi_{n,\mathsf{C}}(T)$.

3. Main results

Let us first recall some notation and results which we will use. Throughout, $\mathbb{M}_n(\mathbb{C})$ is a ring of $n \times n$ matrices over the complex numbers and r is an arbitrary complex number that we shall commonly assume to be nonzero. Any n-th root of r and any primitive n-th root of unity are denoted by $r^{1/n}$ and ω_n, respectively. We use symbols $\{\lambda_j\}_{j=0}^{n-1}$ and $|A|$ to stand for the eigenvalues and the determinant of $A \in \mathbb{M}_n(\mathbb{C})$, respectively. For a given matrix $C = [c_{ij}] \in \mathbb{M}_n(\mathbb{C})$, its *Frobenius norm*, sometimes also called the *Euclidean norm* $\|C\|_E$ and *spectral norm* $\|C\|_2$, are defined by

$$\|C\|_E := \left(\sum_{1 \leqslant i,j \leqslant n} |c_{ij}|^2\right)^{1/2} \quad \text{and} \quad \|C\|_2 := \left(\max_{0 \leqslant j \leqslant n-1} \lambda_j(C^*C)\right)^{1/2},$$

where C^* is the conjugate transpose of C. Note that C^*C is a Hermitian, positive semi-definite matrix, and $\|C\|_2 = \sigma_{\max}(C)$ is the largest singular value of C. There is a well-known inequality between these norms:

$$\frac{1}{\sqrt{n}}\|C\|_E \leqslant \|C\|_2 \leqslant \|C\|_E\,. \tag{15}$$

Given any two $m \times n$ matrices A and B, the *Hadamard product* (also known as the *Schur product*) $C = A \circ B$, is the componentwise product of matrices A and B. That is to say, the elements of $m \times n$ matrix C are given by $c_{ij} = a_{ij}b_{ij}$. It is well known [15, Theorem 1.2.] that

$$\|A \circ B\|_2 \leqslant r_1(A)c_1(B), \tag{16}$$

where $r_1(A)$ is the maximum row length norm of A, and $c_1(B)$ is the maximum column length norm of B, i.e.,

$$r_1(A) = \max_{1 \leqslant i \leqslant m} \sqrt{\sum_{j=1}^{n} |a_{ij}|^2} \quad \text{and} \quad c_1(B) = \max_{1 \leqslant j \leqslant n} \sqrt{\sum_{i=1}^{m} |b_{ij}|^2}\,. \tag{17}$$

THEOREM 1. *Let $n \geqslant 2$ be an integer, $C = \mathrm{Circ}_r(\mathsf{u}_0, \mathsf{u}_1, \mathsf{u}_2, \ldots, \mathsf{u}_{n-1})$ and*

$$P_{n-1} = \sum_{i=0}^{n-1} \mathsf{u}_i^2 = \frac{1}{3}\left((2a+2b-c)^2 - 2a(5b-c) + 2\mathsf{u}_{n-1}\mathsf{u}_n + 4\mathsf{u}_{n-2}\mathsf{u}_n + 4\mathsf{u}_{n-1}\mathsf{u}_{n+1} - \mathsf{u}_{n+2}^2\right).$$

Then we obtain the following estimations:

$$1^\circ)\ \textit{If } |r| \geqslant 1,\ \textit{then}\ \sqrt{P_{n-1}} \leqslant \|C\|_2 \leqslant \sqrt{(|r|^2(P_{n-1} - a^2) + 1)P_{n-1}}\,. \tag{18}$$

$$2^\circ)\ \textit{If } |r| < 1,\ \textit{then}\ |r|\sqrt{P_{n-1}} \leqslant \|C\|_2 \leqslant \sqrt{nP_{n-1}}\,. \tag{19}$$

Proof. $1^\circ)$ Suppose that $|r| \geqslant 1$. Then, we have

$$\|C\|_E^2 = \sum_{i=0}^{n-1}(n-i)\mathsf{u}_i^2 + \sum_{i=1}^{n-1} i|r|^2\mathsf{u}_i^2 \geqslant \sum_{i=0}^{n-1}(n-i)\mathsf{u}_i^2 + \sum_{i=1}^{n-1} i\mathsf{u}_i^2 = n\sum_{i=0}^{n-1}\mathsf{u}_i^2\,.$$

So, $\|C\|_E/\sqrt{n} \geqslant \sqrt{\sum_{i=0}^{n-1} \mathsf{u}_i^2}$, which implies $\|C\|_2 \geqslant \sqrt{\sum_{i=0}^{n-1} \mathsf{u}_i^2} = \sqrt{P_{n-1}}$. For the second inequality note that $C = A \circ B$, where A and B are the following $n \times n$ matrices:

$$A = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ r\mathsf{u}_{n-1} & 1 & 1 & \cdots & 1 & 1 \\ r\mathsf{u}_{n-2} & r\mathsf{u}_{n-1} & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ r\mathsf{u}_2 & r\mathsf{u}_3 & r\mathsf{u}_4 & \cdots & 1 & 1 \\ r\mathsf{u}_1 & r\mathsf{u}_2 & r\mathsf{u}_3 & \cdots & r\mathsf{u}_{n-1} & 1 \end{bmatrix}, \quad B = \begin{bmatrix} \mathsf{u}_0 & \mathsf{u}_1 & \mathsf{u}_2 & \cdots & \mathsf{u}_{n-2} & \mathsf{u}_{n-1} \\ 1 & \mathsf{u}_0 & \mathsf{u}_1 & \cdots & \mathsf{u}_{n-3} & \mathsf{u}_{n-2} \\ 1 & 1 & \mathsf{u}_0 & \cdots & \mathsf{u}_{n-4} & \mathsf{u}_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & \mathsf{u}_0 & \mathsf{u}_1 \\ 1 & 1 & 1 & \cdots & 1 & \mathsf{u}_0 \end{bmatrix}.$$

Then we get

$$r_1(A) = \max_{1 \leqslant i \leqslant n} \sqrt{\sum_{j=1}^{n} |a_{ij}|^2} = \sqrt{\sum_{j=1}^{n} |a_{nj}|^2} = \sqrt{|r|^2 \sum_{i=1}^{n-1} \mathsf{u}_i^2 + 1} = \sqrt{|r|^2 \sum_{i=0}^{n-1} \mathsf{u}_i^2 + 1 - |r|^2 a^2},$$

$$c_1(B) = \max_{1 \leqslant j \leqslant n} \sqrt{\sum_{i=1}^{n} |b_{ij}|^2} = \sqrt{\sum_{i=1}^{n} |b_{in}|^2} = \sqrt{\sum_{i=0}^{n-1} \mathsf{u}_i^2}.$$

Therefore, applying Proposition 2 and inequality (16), we obtain

$$\|C\|_2 \leqslant \sqrt{|r|^2 \sum_{i=0}^{n-1} \mathsf{u}_i^2 + 1 - |r|^2 a^2} \sqrt{\sum_{i=0}^{n-1} \mathsf{u}_i^2} = \sqrt{(|r|^2 (P_{n-1} - a^2) + 1) P_{n-1}}.$$

2°) If $|r| < 1$, we may proceed in a similar way. Then we have

$$\|C\|_E^2 \geqslant \sum_{i=0}^{n-1} (n-i) |r|^2 \mathsf{u}_i^2 + \sum_{i=1}^{n-1} i |r|^2 \mathsf{u}_i^2 = n |r|^2 \sum_{i=0}^{n-1} \mathsf{u}_i^2.$$

Thus, $\|C\|_E/\sqrt{n} \geqslant |r| \sqrt{\sum_{i=0}^{n-1} \mathsf{u}_i^2}$, which implies $\|C\|_2 \geqslant |r| \sqrt{\sum_{i=0}^{n-1} \mathsf{u}_i^2} = |r| \sqrt{P_{n-1}}$.

Let us present the matrix C as a Hadamard product $C = A \circ B$, where A and B are matrices defined by

$$A = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ r & 1 & 1 & \cdots & 1 & 1 \\ r & r & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ r & r & r & \cdots & 1 & 1 \\ r & r & r & \cdots & r & 1 \end{bmatrix}, \quad B = \begin{bmatrix} \mathsf{u}_0 & \mathsf{u}_1 & \mathsf{u}_2 & \cdots & \mathsf{u}_{n-2} & \mathsf{u}_{n-1} \\ \mathsf{u}_{n-1} & \mathsf{u}_0 & \mathsf{u}_1 & \cdots & \mathsf{u}_{n-3} & \mathsf{u}_{n-2} \\ \mathsf{u}_{n-2} & \mathsf{u}_{n-1} & \mathsf{u}_0 & \cdots & \mathsf{u}_{n-4} & \mathsf{u}_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathsf{u}_2 & \mathsf{u}_3 & \mathsf{u}_4 & \cdots & \mathsf{u}_0 & \mathsf{u}_1 \\ \mathsf{u}_1 & \mathsf{u}_2 & \mathsf{u}_3 & \cdots & \mathsf{u}_{n-1} & \mathsf{u}_0 \end{bmatrix}.$$

Then we obtain

$$r_1(A) = \max_{1 \leqslant i \leqslant n} \sqrt{\sum_{j=1}^{n} |a_{ij}|^2} = \sqrt{\sum_{j=1}^{n} |a_{1j}|^2} = \sqrt{\sum_{j=1}^{n} 1} = \sqrt{n},$$

$$c_1(B) = \max_{1 \leqslant j \leqslant n} \sqrt{\sum_{i=1}^{n} |b_{ij}|^2} = \sqrt{\sum_{i=1}^{n} |b_{i1}|^2} = \sqrt{\sum_{i=0}^{n-1} \mathsf{u}_i^2}\,.$$

According to (16), we have $\|C\|_2 \leqslant \sqrt{n \sum_{i=0}^{n-1} \mathsf{u}_i^2} = \sqrt{nP_{n-1}}$, which completes the proof. □

Clearly, the previous formulas can be easily modified to the case

$$C = \mathrm{Circ}_r(\mathsf{u}_k, \mathsf{u}_{k+1}, \dots, \mathsf{u}_{k+n-1}),$$

for $k \geqslant 1$. Let us emphasize the obtained result with the following example.

EXAMPLE 1. Suppose that $C_r = \mathrm{Circ}_r(\mathcal{N}_0, \mathcal{N}_1, \mathcal{N}_2, \dots, \mathcal{N}_{n-1})$ and let us denote by $L(|r|,n)$ and $R(|r|,n)$ the lower and upper bounds for the spectral norm $\|C_r\|_2$, respectively. According to (18) and (19) we have

$$[L(|r|,n), R(|r|,n)] = \begin{cases} \left[\sqrt{P_{n-1}}, \sqrt{(|r|^2 P_{n-1}+1)P_{n-1}}\right], & |r| \geqslant 1, \\ [\,|r|\sqrt{P_{n-1}}, \sqrt{nP_{n-1}}\,], & |r| < 1. \end{cases}$$

The following table illustrates some bounds for the largest singular value $\sigma_{\max}(C_r)$ for $|r| = 1/2$, $|r| = 1$ and $|r| = 2$.

n	$[L(1,n), R(1,n)]$	$[L(1/2,n), R(1/2,n)]$	$[L(2,n), R(2,n)]$
2	$[1, \sqrt{2}] \approx [1, 1.41]$	$[\frac{1}{2}, \sqrt{2}] \approx [0.5, 1.41]$	$[1, \sqrt{5}] \approx [1, 2.24]$
3	$[\sqrt{2}, \sqrt{6}] \approx [1.41, 2.45]$	$[\frac{1}{2}\sqrt{2}, \sqrt{6}] \approx [0.71, 2.45]$	$[\sqrt{2}, 3\sqrt{2}] \approx [1.41, 4.24]$
4	$[\sqrt{3}, 2\sqrt{3}] \approx [1.73, 3.46]$	$[\frac{1}{2}\sqrt{3}, 2\sqrt{3}] \approx [0.87, 3.46]$	$[\sqrt{3}, \sqrt{39}] \approx [1.73, 6.25]$
5	$[\sqrt{7}, 2\sqrt{14}] \approx [2.65, 7.48]$	$[\frac{1}{2}\sqrt{7}, \sqrt{35}] \approx [1.32, 5.92]$	$[\sqrt{7}, \sqrt{203}] \approx [2.65, 14.25]$
6	$[4, 4\sqrt{17}] \approx [4, 16.49]$	$[2, 4\sqrt{6}] \approx [2, 9.80]$	$[4, 4\sqrt{65}] \approx [4, 32.25]$
7	$[4\sqrt{2}, 4\sqrt{33}] \approx [5.66, 22.98]$	$[2\sqrt{2}, 4\sqrt{14}] \approx [2.83, 14.97]$	$[4\sqrt{2}, 4\sqrt{258}] \approx [5.66, 64.25]$
8	$[7\sqrt{2}, 21\sqrt{22}] \approx [9.90, 98.50]$	$[\frac{7}{2}\sqrt{2}, 28] \approx [4.95, 28]$	$[7\sqrt{2}, 7\sqrt{786}] \approx [9.90, 196.25]$

Table 2: *Upper and lower bounds for* $\sigma_{\max}(C_r)$, $|r| \in \{1, 1/2, 2\}$.

Let us see how accurate these estimates are. If $r = e^{i\varphi}$, consider for example $\varphi = \pi/2$ and the matrix $C_i = \mathrm{Circ}_i(\mathcal{N}_0, \mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3, \mathcal{N}_4, \mathcal{N}_5)$, i.e.,

$$C_i = \begin{bmatrix} 0 & 1 & 1 & 1 & 2 & 3 \\ 3i & 0 & 1 & 1 & 1 & 2 \\ 2i & 3i & 0 & 1 & 1 & 1 \\ i & 2i & 3i & 0 & 1 & 1 \\ i & i & 2i & 3i & 0 & 1 \\ i & i & i & 2i & 3i & 0 \end{bmatrix}.$$

A direct calculation shows that $\sigma_{\max}(C_i) = (10\sqrt{2} + 6\sqrt{3} + 5\sqrt{6} + 19)^{1/2} \approx 7.48$. Similarly, for matrices $C_{1/2} = \mathrm{Circ}_{1/2}(0,1,1,1,2,3)$ and $C_2 = \mathrm{Circ}_2(0,1,1,1,2,3)$ one can find $\sigma_{\max}(C_{1/2}) \approx 6.03$ and $\sigma_{\max}(C_2) \approx 13.52$. Note that previous example suggests that $\sigma_{\max}(C_r)$ can be expected to be close to the $(L(|r|,n)+R(|r|,n))/2$.

In order to determine the eigenvalues of the matrix $C = \mathrm{Circ}_r(\mathsf{u}_0, \mathsf{u}_1, \mathsf{u}_2, \ldots, \mathsf{u}_{n-1})$, we need the following result.

LEMMA 2. ([9], Lemma 4) *Let $C = \mathrm{Circ}_r(c_0, c_1, c_2, \ldots, c_{n-1})$ be an r-circulant matrix. Then the eigenvalues of C are*

$$\lambda_j = \sum_{i=0}^{n-1} c_i (r^{1/n}\omega_n^{-j})^i, \quad j = 0,1,\ldots,n-1, \tag{20}$$

where $r^{1/n}$ is any n-th root of r and ω_n is any primitive n-th root of unity.

THEOREM 2. *Let $n \geqslant 2$ be an integer. Suppose that α, β, γ are the roots of the characteristic equation $x^3 - x^2 - 1 = 0$, and*

$$X = \frac{a\beta\gamma - b(\beta+\gamma)+c}{(\alpha-\gamma)(\alpha-\beta)}, \quad Y = -\frac{a\alpha\gamma - b(\alpha+\gamma)+c}{(\beta-\gamma)(\alpha-\beta)}, \quad Z = \frac{a\alpha\beta - b(\alpha+\beta)+c}{(\alpha-\gamma)(\beta-\gamma)}.$$

Then, the eigenvalues $\{\lambda_j\}_{j=0}^{n-1}$, of the matrix $C = \mathrm{Circ}_r(\mathsf{u}_0, \mathsf{u}_1, \mathsf{u}_2, \ldots, \mathsf{u}_{n-1})$ are given by the following formulas:

1°) *If $r^{1/n}\omega_n^{-j} \notin \{1/\alpha, 1/\beta, 1/\gamma\}$, then*

$$\lambda_j = \frac{a - r\mathsf{u}_n - (a-b+r\mathsf{u}_{n-2})r^{1/n}\omega_n^{-j} - (b-c+r\mathsf{u}_{n-1})r^{2/n}\omega_n^{-2j}}{1 - r^{1/n}\omega_n^{-j} - r^{3/n}\omega_n^{-3j}}.$$

2°) *If $r^{1/n}\omega_n^{-j} = 1/\alpha$, then* $\lambda_j = nX + \dfrac{\alpha^n - \beta^n}{\alpha^{n-1}(\alpha-\beta)}Y + \dfrac{\alpha^n - \gamma^n}{\alpha^{n-1}(\alpha-\gamma)}Z.$

3°) *If $r^{1/n}\omega_n^{-j} = 1/\beta$, then* $\lambda_j = \dfrac{\beta^n - \alpha^n}{\beta^{n-1}(\beta-\alpha)}X + nY + \dfrac{\beta^n - \gamma^n}{\beta^{n-1}(\beta-\gamma)}Z.$

4°) *If $r^{1/n}\omega_n^{-j} = 1/\gamma$, then* $\lambda_j = \dfrac{\gamma^n - \alpha^n}{\gamma^{n-1}(\gamma-\alpha)}X + \dfrac{\gamma^n - \beta^n}{\gamma^{n-1}(\gamma-\beta)}Y + nZ.$

Proof. 1°) Note that $x^3 + x - 1 = (x-\alpha^{-1})(x-\beta^{-1})(x-\gamma^{-1})$. Therefore, if $r^{1/n}\omega_n^{-j} \notin \{\alpha^{-1}, \beta^{-1}, \gamma^{-1}\}$, then $r^{1/n}\omega_n^{-j}$ is not the root of the polynomial $x^3 + x - 1$.

Hence, using Lemma 2 and Proposition 3, we have the following identity:

$$\begin{aligned}\lambda_j &= \sum_{i=0}^{n-1} \mathsf{u}_i r^{i/n} \omega_n^{-ij} = \Phi_{n,\mathsf{u}}(r^{1/n}\omega_n^{-j}) \\ &= \frac{a+(b-a)r^{1/n}\omega_n^{-j}+(c-b)(r^{1/n}\omega_n^{-j})^2}{1-r^{1/n}\omega_n^{-j}-(r^{1/n}\omega_n^{-j})^3} \\ &\quad - \frac{\mathsf{u}_n(r^{1/n}\omega_n^{-j})^n+\mathsf{u}_{n-2}(r^{1/n}\omega_n^{-j})^{n+1}+\mathsf{u}_{n-1}(r^{1/n}\omega_n^{-j})^{n+2}}{1-r^{1/n}\omega_n^{-j}-(r^{1/n}\omega_n^{-j})^3} \\ &= \frac{a-r\mathsf{u}_n-(a-b+r\mathsf{u}_{n-2})r^{1/n}\omega_n^{-j}-(b-c+r\mathsf{u}_{n-1})r^{2/n}\omega_n^{-2j}}{1-r^{1/n}\omega_n^{-j}-(r^{1/n}\omega_n^{-j})^3}.\end{aligned}$$

2°) Previously we have observed that $\mathsf{u}_n = \alpha^n X + \beta^n Y + \gamma^n Z$. Therefore, by Lemma 2, we have the following identity

$$\begin{aligned}\lambda_j &= \sum_{i=0}^{n-1} \mathsf{u}_i r^{i/n}\omega_n^{-ij} \\ &= X\sum_{i=0}^{n-1}(\alpha r^{1/n}\omega_n^{-j})^i + Y\sum_{i=0}^{n-1}(\beta r^{1/n}\omega_n^{-j})^i + Z\sum_{i=0}^{n-1}(\gamma r^{1/n}\omega_n^{-j})^i.\end{aligned} \tag{21}$$

Assume that $r^{1/n}\omega_n^{-j} = 1/\alpha$. From (21), it follows that

$$\begin{aligned}\lambda_j &= X\sum_{i=0}^{n-1}\left(\frac{\alpha}{\alpha}\right)^i + Y\sum_{i=0}^{n-1}\left(\frac{\beta}{\alpha}\right)^i + Z\sum_{i=0}^{n-1}\left(\frac{\gamma}{\alpha}\right)^i \\ &= nX + \frac{\alpha^n-\beta^n}{\alpha^{n-1}(\alpha-\beta)}Y + \frac{\alpha^n-\gamma^n}{\alpha^{n-1}(\alpha-\gamma)}Z.\end{aligned}$$

Similarly, if $r^{1/n}\omega_n^{-j} = 1/\beta$ or $r^{1/n}\omega_n^{-j} = 1/\gamma$, we can proceed in the same way as in the second case to obtain 3°) and 4°). Thus, the theorem is proved. □

Research papers [5], [8] and [18] motivate us to move on to some questions about the invertibility of the introduced matrices. Using Theorem 2, one can give some partial results on the invertibility of circulant and skew circulant matrices involving the generalized Narayana numbers. First we need the following proposition.

PROPOSITION 4. *The matrix* $C = \mathrm{Circ}_r(\mathsf{u}_0, \mathsf{u}_1, \mathsf{u}_2, \dots, \mathsf{u}_{n-1})$ *is invertible if and only if*

$$\gcd(\Phi_{n,\mathsf{u}}(t), t^n - r) = 1.$$

Proof. Let $C = \mathrm{Circ}_r(\mathsf{u}_0, \mathsf{u}_1, \dots, \mathsf{u}_{n-1})$ and assume that $\{\lambda_0, \lambda_1, \dots, \lambda_{n-1}\}$ are the eigenvalues of C. Then C is singular iff $\lambda_j = 0$ for some $j \in \{0, 1, \dots, n-1\}$.

According to Lemma 2, the j-th eigenvalue of the matrix C is equal to

$$\lambda_j = \sum_{k=0}^{n-1} \mathsf{u}_k (r^{1/n}\omega_n^{-j})^k = \Phi_{n,\mathsf{u}}(r^{1/n}\omega_n^{-j}),$$

where $r^{1/n}$ is any n-th root of r and ω_n is any primitive n-th root of unity. Therefore, the matrix C is singular if and only if $\Phi_{n,\mathsf{u}}(r^{1/n}\omega_n^{-j}) = 0$ for some $j \in \{0,1,\dots,n-1\}$. Since

$$(r^{1/n}\omega_n^{-j})^n = r,$$

$r^{1/n}\omega_n^{-j}$ is the root of the polynomial $t^n - r$, and we are done. □

THEOREM 3. *Let m be any positive integer. The circulant matrix*

$$C = \mathrm{Circ}(\mathsf{u}_0, \mathsf{u}_1, \dots, \mathsf{u}_{n-1})$$

is singular if and only if C has one of the following forms:

$$\begin{aligned}
&\mathrm{Circ}(m,m), \quad \mathrm{Circ}(m,m,m), \quad \mathrm{Circ}(m,0,m,2m),\\
&\mathrm{Circ}(m,2m,m,2m), \quad \mathrm{Circ}(m,0,0,m,m,m),\\
&\mathrm{Circ}(4m,2m,m,5m,7m,8m) \quad \textit{or} \quad \mathrm{Circ}(0,m,0,0,m,m,m,2m).
\end{aligned}$$

Proof. Obviously, $\mathrm{Circ}(a,b)$ is singular if and only if $a = b$, so let us suppose that $n \geqslant 3$. Since $r = 1$, we can take $r^{1/n} = \omega_n^k$, for some positive integer k, $k < n$. According to Theorem 2, we have that the j-th eigenvalue λ_j, for $j = 0,1,\dots,n-1$, is equal to

$$\lambda_j = \Phi_{n,\mathsf{u}}(\omega^{k-j}) = \frac{a - \mathsf{u}_n + (b - a - \mathsf{u}_{n-2})\omega_n^{k-j} + (c - b - \mathsf{u}_{n-1})\omega_n^{2(k-j)}}{1 - \omega_n^{k-j} - \omega_n^{3(k-j)}}.$$

Since $|C| = \prod_{i=0}^{n-1} \lambda_j$, the matrix C is singular if and only if

$$a - \mathsf{u}_n + (b - a - \mathsf{u}_{n-2})\omega_n^{k-j} + (c - b - \mathsf{u}_{n-1})\omega_n^{2(k-j)} = 0 \tag{22}$$

for some $j = 0,1,\dots,n-1$. Let us consider the following cases:

1. If $k \neq j$, then ω_n^{k-j} is the complex number such that $|\omega_n^{k-j}| = 1$. Then it should be $a - \mathsf{u}_n = c - b - \mathsf{u}_{n-1}$, that is $a + b - c = \mathsf{u}_{n-3}$. If $n \geqslant 10$, then $\mathsf{u}_{n-3} \neq a + b - c$ since
$$a(\mathscr{N}_{n-5} - 1) + b(\mathscr{N}_{n-6} - 1) + c(\mathscr{N}_{n-4} + 1) > 0. \tag{23}$$
Namely, expressions in parentheses are positive integers, while a, b and c are nonnegative integers such that $a^2 + b^2 + c^2 \neq 0$.

   If $3 \leqslant n < 10$, one can easily examine all cases such that $a + b - c = \mathsf{u}_{n-3}$. Taking into account that a, b and c are nonnegative integers such that $a^2 + b^2 + c^2 \neq 0$, it is straightforward to check the following:

   1.1. $\mathrm{Circ}(a,b,b)$ is singular if and only if $a = b$.

   1.2. $\mathrm{Circ}(a,b,a,2a)$ is singular if and only if $b = 0$ or $b = 2a$.

   1.3. $\mathrm{Circ}(a,2c-a,c,a+c,3c)$ is regular for all a and c.

1.4. $\mathrm{Circ}(a,2c,c,a+c,a+3c,a+4c)$ is singular if and only if $a=4c$ or $c=0$.
1.5. $\mathrm{Circ}(a,b,0,a,a+b,a+b,2a+b)$ is regular for all a and b.
1.6. $\mathrm{Circ}(a,b,0,a,a+b,a+b,2a+b,3a+2b)$ is singular if and only if $a=0$.
1.7. $\mathrm{Circ}(0,b,0,0,b,b,b,2b,3b)$ is regular for all b.

2. If $k=j$, then $\omega_n^{j-k}=1$. Therefore, the equality (22) reduces to

$$a-\mathsf{u}_n+b-a-u_{n-2}+c-b-\mathsf{u}_{n-1}=0,$$

or equivalently $\mathsf{u}_{n+2}=c$. Thus, we obtain

$$a\mathscr{N}_n+b\mathscr{N}_{n-1}+c(\mathscr{N}_{n+1}-1)=0,$$

which is impossible. □

From the theorem just proved, we get as an immediate consequence the result on the invertibility of circulant matrices involving the ordinary Narayana numbers.

COROLLARY 3. *The circulant matrix*

$$C=\mathrm{Circ}(\mathscr{N}_0,\mathscr{N}_1,\mathscr{N}_2,\ldots,\mathscr{N}_{n-1})$$

is invertible for all $n\geqslant 2$.

THEOREM 4. *The skew circulant matrix* $C=\mathrm{Circ}_{-1}(\mathsf{u}_0,\mathsf{u}_1,\ldots,\mathsf{u}_{n-1})$ *is singular if and only if the matrix* C *is of the form* $C=\mathrm{Circ}_{-1}(0,m,m)$ *for some positive integer* m.

Proof. Clearly, $\mathrm{Circ}_{-1}(a,b)$ is regular for all a and b (we assume $a^2+b^2\neq 0$), so let us suppose that $n\geqslant 3$. Since $r=-1$, we can take $r^{1/n}=\omega_n^{k/2}$, for some positive integer k, $k<n$. According to Theorem 2, we have that the j-th eigenvalue, $j=0,1,\ldots,n-1$, is equal to

$$\lambda_j=\Phi_{n,\mathsf{u}}(\omega^{k/2-j})=\frac{a+\mathsf{u}_n+(b-a+\mathsf{u}_{n-2})\omega_n^{k/2-j}+(c-b+\mathsf{u}_{n-1})\omega_n^{2(k/2-j)}}{1-\omega_n^{k/2-j}-\omega_n^{3(k/2-j)}}.$$

Since $|C|=\prod_{i=0}^{n-1}\lambda_j$, the matrix C will be singular if and only if

$$a+\mathsf{u}_n+(b-a+\mathsf{u}_{n-2})\omega_n^{k/2-j}+(c-b+\mathsf{u}_{n-1})\omega_n^{2(k/2-j)}=0,$$

for some $j=0,1,\ldots,n-1$. Let us consider the following cases:

1. If $k/2\neq j$, then $\omega_n^{k/2-j}$ is the complex number such that $|\omega_n^{k/2-j}|=1$. Then it should be $a+\mathsf{u}_n=c-b+\mathsf{u}_{n-1}$, i.e., $c-a-b=\mathsf{u}_{n-3}$.

If $n \geqslant 7$, along the same argument as in (23), we see that

$$a(\mathcal{N}_{n-5}+1)+b(\mathcal{N}_{n-6}+1)+c(\mathcal{N}_{n-4}-1)>0.$$

Therefore, $\mathsf{u}_{n-3} \neq c-a-b$.

If $3 \leqslant n < 7$, one can quickly check the remaining cases assuming that a, b and c are nonnegative integers such that $a^2+b^2+c^2 \neq 0$.

1.1. $\mathrm{Circ}_{-1}(a,b,2a+b)$ is singular if and only if $a=0$.
1.2. $\mathrm{Circ}_{-1}(a,b,a+2b,2a+2b)$ is regular for all a and b.
1.3. $\mathrm{Circ}_{-1}(0,0,c,c,c)$ is regular for all c.
1.4. $\mathrm{Circ}_{-1}(0,0,c,c,c,2c)$ is regular for all c.

2. If $k=2j$, then $\omega_n^{k/2-j}=1$. Hence, C is singular if and only if

$$a+\mathsf{u}_n+b-a+u_{n-2}+c-b+\mathsf{u}_{n-1}=0 \Leftrightarrow \mathsf{u}_{n+2}=-c$$

i.e., $a\mathcal{N}_n+b\mathcal{N}_{n-1}+c(\mathcal{N}_{n+1}+1)=0$, which is impossible. $\square$

As a consequence of the previous theorem, we get the following interesting result on the invertibility of skew circulant matrices involving the ordinary Narayana numbers.

COROLLARY 4. *Let $n \geqslant 2$ be an integer. The skew circulant matrix*

$$C=\mathrm{Circ}_{-1}(\mathcal{N}_0,\mathcal{N}_1,\mathcal{N}_2,\ldots,\mathcal{N}_{n-1})$$

is invertible if and only if $n \neq 3$.

4. Discussion

If $r^{1/n}\omega_n^{-j} \notin \{1/\alpha, 1/\beta, 1/\gamma\}$, in order to conclude whether the given matrix $C_r = \mathrm{Circ}_r(\mathsf{u}_0,\mathsf{u}_1,\mathsf{u}_2,\ldots,\mathsf{u}_{n-1})$ is singular or not, we are referred to the equation

$$a-r\mathsf{u}_n-(a-b+r\mathsf{u}_{n-2})r^{1/n}\omega_n^{-j}-(b-c+r\mathsf{u}_{n-1})r^{2/n}\omega_n^{-2j}=0. \tag{24}$$

As can be seen, the general case is quite difficult. Basically, except for the members of the Narayana sequence, (24) depends only on the initial conditions and the parameter r. This provides the possibility to obtain an invertible matrix by varying the initial conditions.

In Theorems 3 and 4, we have discussed special cases for $r=1$ and $r=-1$. Besides, any single case can be considered separately using equation (24), which can produce notable interesting results on the invertibility of such matrices.

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