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H-coloring revisited

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Abstract. In this paper we give a new, shortened proof of NP-completeness of CSP problem for undirected, non bipartite graphs, of interest for generalization to QCSP problem. We also give some illustrative examples.

1. Introduction

In the past 30 years the constraint satisfaction problem (CSP) has seen a lot of interest (see for instance [1, 2, 6, 9, 10]). The proof of the dichotomy conjecture ([12] and [5]) has recently been published. Recently, quantified constrained satisfaction problem (QCSP) has seen growing interest ([3, 4, 7, 8, 11]). As in the case of CSP in the past, QCSP is being considered first for the simpler cases, like semicomplete graphs (the case was completely solved in [8]), and the case of undirected graphs is yet to be understood. For this reason, it makes sense to revisit the original proof of Hell and Nešetřil for the dichotomy for CSP for undirected graphs ([9]). In this paper we give new simpler proof of central theorem from [9]. Our proof is simpler, as it considers fewer cases, and is perhaps more conducive to generalizations to QCSP. We also examine a few related, illustrative examples.

As usually in this area, all graphs we consider are finite and with no multiple edges.

We are going to use several preliminary results and constructions from [9].

Proposition 1.1. *Let H_1 be a sub-graph of the graph H. Supposing that there is endomorphism e, $e : H \to H_1$, then for the each graph G there is homomorphism h, $h : G \to H$, if and only if there is homomorphism h_1 ($h_1 : G \to H_1$).*

The graph H such that there is no endomorphism from H into itself other than an automorphism is called a *core* graph.

H-colouring: Let H and G be two undirected graphs. We say that G has an H-colouring if and only if there is a graph homomorphism from H to G.

Indicator construction: Let I be a fixed graph, and let i and j be distinct vertices of I such that some automorphism of I maps i to j and j to i. This construction transforms graph H into H' so that H' has the same set of vertices as H and the vertices h and h' are connected when there exists a homomorphism $f : I \to H$ which i maps to h and j maps to h'.

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Lemma 1.2. *Let H' graph be obtained from H by indicator construction. If H'-colouring problem NP-complete then the H-colouring is NP-complete.*

The sub-indicator construction: Let J be a fixed graph with specified vertices $j, h_1, h_2...., h_t$. This construction transforms a core graph $H(V, E)$ with t specified vertices $k_1, k_2...., k_t$ into its sub-graph H' whose set of vertices is V' defined as follows: Let W be obtained as the disjoint union of graphs J and H. The set V' is set of all vertices such that there exists a retraction r, $r : W \to H$, such that for each $i \leq t$, $r(h_i) = k_i$.

Lemma 1.3. *Let H' graph be obtained from core-graph H by sub-indicator construction. If H'-colouring problem is NP-complete then the problem of H- colouring is NP-complete.*

The edge sub-indicator construction: Let J be a graph with a specified pair of vertices $\{u, v\}$ and t specified vertices $h_1, h_2, ..., h_t$ such that some automorphism of J keeps each vertices h_i fixed while exchanging u and v. This construction transforms a given core-graph H with t specified vertices $k_1, k_2, ..., k_t$ into a graph H' in the following way: Let W be the disjoint union of J and H. H' is determined by those edges kk' such that there exists a retraction r, $r : W \to H$ such that for each $i \leq t$, $r(h_i) = k_i$ and such that r maps the set $\{u, v\}$ onto $\{k, k'\}$.

Lemma 1.4. *Let H^* graph be obtained from core-graph H by edge sub-indicator construction. If H^*-colouring problem is NP-complete than the problem of H- colouring is NP-complete.*

Note that the indicator and sub-indicator can be understood as a form of definition. The definitions that use only primitive-positive formulas i.e. only with existential quantifiers with conjunction of positive atomic formulas under quantification, are the ones that are allowed and graphs defined using such formulas can also be obtained using indicator, the sub-indicator constructions and the edge sub-indicator constructions from the paper of Hell and Nešetřil.

2. The new proof

In this section we give a new, shorter proof of the main theorem of Hell and Nešetřil. The result is that H-colouring is NP-complete iff H is not bipartite. If H happens to be bipartite, then the H-colouring is clearly tractable. It remains to prove that for any H which is not bipartite, the H-colouring problem is *NP*-complete.

We denote by $\mathcal{K}$ the class of all graphs H which are not bipartite and such that the H-colouring problem is NP-complete. If $\mathcal{K}$ is non-empty, then we fix a graph $H^* \in \mathcal{K}$ with the fewest number of vertices, n, and of all graphs in $\mathcal{K}$ with n vertices, H^* has the most edges.

In Hell and Nešetřil's paper [9], the following properties were proved about H^*:

Lemma 2.1. *0) H^* is a core.*
1) H^* does not contain K_4.
2) H^* contains a triangle.
3) Each two vertices of H^* have a common neighbour (there is a path length 2 between any two vertices).
4) H^* does not contain K_4^-.

Proof. 0) By Proposition 1.1.
1) and 3) Assuming that H^* fails to have one of these two properties, Hell and Nešetřil use sub-indicator constructions to find a graph in $\mathcal{K}$ with fewer vertices than H^*.
2) Similar to 1) and 3) Hell and Nešetřil use an indicator construction to obtain a graph in $\mathcal{K}$ with n vertices and more edges than H^*.

4) Consider the graph S (Figure 2). We first prove that there is no homomorphism from S to H^*. Assume that there exists such a homomorphism and call the images of $u, v, w, \dots$ by $u', v', w', \dots$. We apply the sub-indicator construction with indicator J (Figure 2). If we apply this construction with $k_1 = u$ and $k_2 = v$, we obtain a non-bipartite graph which contains the triangle a', b', c', but does not contain vertex w' (otherwise H^* would contain K_4). But this would contradict the minimality of the graph H^*, a contradiction, so there can be no homomorphism from S to H^*.

We assume that K_4^- where u and v are non neighbour vertices is a sub-graph of H^*. We apply the indicator construction with the indicator I (Figure 1) on H^*. As in Hell and Nešetřil's proof, we would obtain a graph in $\mathcal{K}$ with the same number n of vertices, but more edges than H^*.

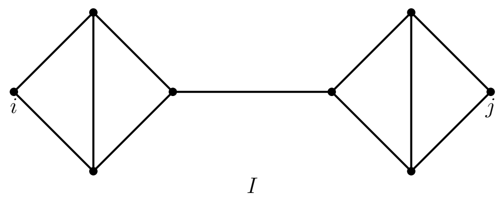


Figure 1: Graph I.

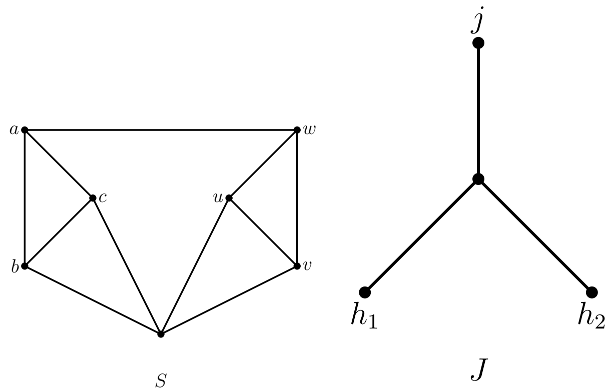


Figure 2: Graph S and indicator J.

□

Theorem 2.2. *H^* is 3-colourable.*

Proof. Let us observe any arbitrary vertex of H^* and call it the central vertex.

We say that the neighbouring vertices of the central vertex are on the second level in relation to the central vertex. The third level contains vertices which are at a distance of two from the central vertex. The edge is on the second level if its incident vertices are on the second level. The edge is on the third level if its incident vertices are on the third level. Any vertex of the graph can be considered to be a central vertex, which will be used in following analysis. If several triangles have a common vertex, such triangles are called *petals*. Since H^* has no sub-graph K_4^- nor K_4, the induced sub-graph of H^* on the central vertex and the vertices on the second level consists only of petals. Consequently, the induced sub-graph on the second level is just a 1-factor, i.e. a disjoint union of edges with no isolated vertices (since each second-level vertex has a path of length 2 to the central vertex).

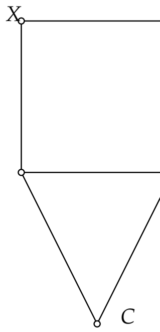


Figure 3: Indicator G_1 used in 1.

Lemma 2.3. *H^* has two more properties:*
1. In relation to one chosen central vertex S, for every edge on the third level, there is a square where one of the opposing sides is contained in the second level and the other is that edge on the third level.
2. There is a vertex X on the third level that has a neighbour in each petal from central vertex (X has neighbour in all edges from second level).

Proof. 1: Let us apply the edge sub-indicator construction. The indicator is the graph G_1 (see Figure 3). The new graph H^{**} obtained from this construction will contain only those edges AB for which there is a homomorphism h, $h : I \to H^*$, mapping C to the given central vertex S of H^*, under which the image of an edge XY is AB. This construction leaves all the edges from the second level, all the incident edges with S as well as the edges which connect one vertex from the second level with one vertex from the third level. We remove only the edges from the third level for which we do not have an edge in the second level so that the two of them are the opposite sides of square. Assume that H^{**} has fewer edges than H^*, say MN is an edge of H^* and not of H^{**}. By Lemma 2.1 3), there exists some vertex P such that $M - P - N$ in H^*. Thus $P - M - N$ is a path of length 2 in H^* connecting P and N. By Lemma 2.1 1) and 4), there can be no vertex $M' \neq M$ such that $P - M' - N$ is a path in H^*. Thus H^{**} has no length 2 path from P to N. Of course, $H^{**} \in \mathcal{K}$. By the proof of Lemma 2.1 3), there is a graph in $\mathcal{K}$ with fewer vertices than H^{**}, contradicting the choice of H^*.

2: We achieve this property in four steps.

Step 1:
According to the proof of 1., when we consider any vertex such that the central vertex, each edge in the third level has a corresponding edge in the second level so that they are opposite sides of square.

Step 2:
Consider the graph U on the Figure 4. For any homomorphism $h : U \to H^*$, i and j are mapped to neighbours.

We apply the indicator construction with the indicator U to H^* and obtain the graph H^{**}. If the obtained graph has a loop then i and j can be mapped to the same vertex. But then, there would exist a homomorphism from the graph S (Figure 2) to H^*, which we proved was impossible in the proof of Lemma 2.1 4).

Consider ab, an edge of H^*. Since there must exist a length 2 path in H^* from a to b, there exists some vertex c of H^* such that a, b and c form a triangle in H^*. Note that U can be mapped to a triangle, but i and j can not map to the same vertex by such a map f. Compose f with the map which maps that triangle onto $\{a, b, c\}$ so that $f(i)$ maps to a and $f(j)$ maps to b. Hence, every edge ab of H^* is also an edge of H^{**}. By the

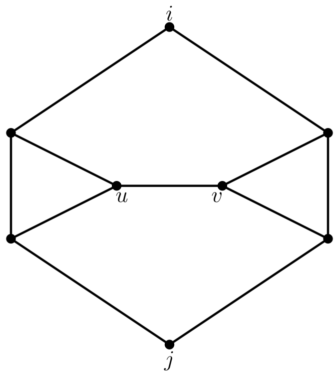


Figure 4: Graph U.

maximality of the number of edges of H^*, $H^{**} = H^*$ and Step 2 is proved.

In the following two steps we prove that there is a vertex on the third level which has a neighbour in each edge from the second level.

Step 3:

Let us consider the vertex X on the third level which has a neighbour in the maximal number of petals. We suppose that there is an edge on the second level whose incident vertices are not neighbours of X. We label this edge CD. According to Lemma 2.1, X is connected to C and D with paths of length two. C and D do not have a common neighbour on the third level, because H^* does not contain K_4^-. Neither C nor D has another neighbour on the second level, as we mentioned above. Hence, the vertex X can connect with C and D through two vertices from the third level, we label those vertices A and B. The two vertices are common neighbours with C and D. If we consider X as a central vertex, the edge CD is on the third level. According to 1., CD has a corresponding edge on the second level so that they are opposite sides of a square. It follows we may assume without a loss of generality that A is connected with B (Figure 5).

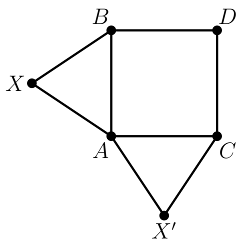


Figure 5: Graph from step 3.

Step 4: If there is a vertex X' which is connected with C or D, according to previous step X' must be on the third level (Figure 6). Let us assume that X is connected to vertex T on the second level. Because of step 2 all homomorphisms from U to the considered graph map i and j to neighbour vertices. We can see that there is a homomorphism from U that u and v map to B and D. At the same homomorphism i and j map to T and X', so these two vertices are connected with an edge. Hence, if there is such an X' then it is a neighbour with all petals which are connected with X, a contradiction with the maximality of X. So it follows that there is vertex on the third level of H^* which has neighbour in all edges from the second level.

□

We can define a 3-colouring of the second level of H^*: the vertices connected with X are coloured in color 1, and the vertices not connected with X are coloured with color 2. We colour the central vertex by

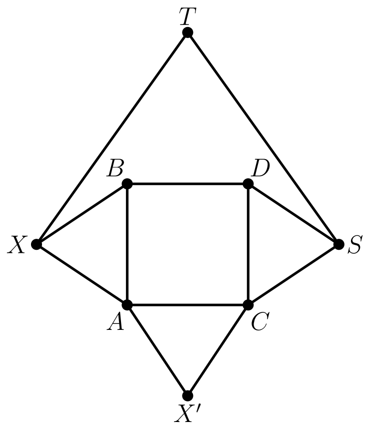


Figure 6: Graph from step 4.

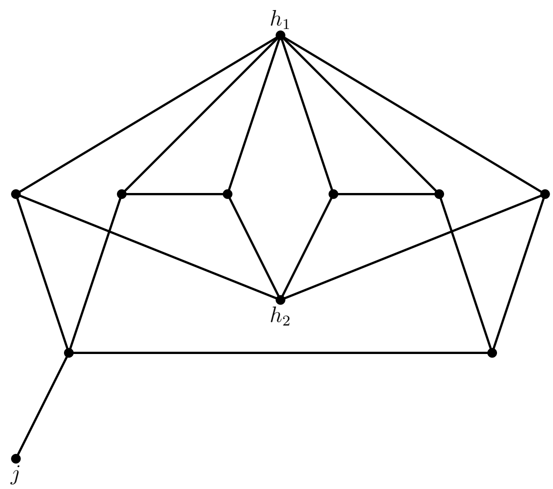


Figure 7: Indicator I_1.

color 3. Clearly, the first and second level have a 3-colouring. We extend this colouring to the third level in the following way: X is coloured by 2 and any vertex on the third level which is connected to a second-level vertex of colour 1 and another second-level vertex of colour 2 is coloured by the colour 3. The remaining third-level vertices can be coloured by 1 and 2, the opposite colour from their second-level neighbours.

It is easy to see that the two vertices one from level two and the other from level three are not joined by an edge if they have the same color. We cannot colour both incident vertices of an edge from the third level in the color 1 or both in the color 2, since an edge from the third level has corresponding edge from the second level and they make the square. Hence, our colouring is either a good 3-colouring of H^* or there are incident vertices of an edge from the third level that are coloured in color 3. In the first case our proof is finished, in the second case let AB be the edge on the third level such that both are coloured by the colour 3. We apply the sub-indicator construction with an indicator I_1 (Figure 7). In the graph H^*, the central vertex is labelled with k_1, and the vertex X is labelled with k_2. We denote the obtained graph by H^{**}

It is easy to check that j cannot map to the central vertex. Moreover, j can map to any neighbour of either A or of B. Thus A and B are both among the vertices of H^{**}. Consider the vertex C such that $A - C - B$

is a length 2 path in H^*. Clearly C is also a vertex of H^{**} and H^{**} has a triangle. Thus H^{**} contains a triangle, so it is not bipartite. Now, $H^{**} \in \mathcal{K}$, but it has fewer vertices than H^* since it does not contain the central vertex. This contradicts the minimality of H^*. □

Corollary 2.4 (Hell and Nešetřil's Theorem). *Given a graph H, the H-colouring problem is NP-complete iff H is not bipartite.*

Proof. By Theorem 2.2, the counterexample H^* is 3-colourable, so it can be mapped onto a triangle. Moreover, by Lemma 2.1 2), H^* contains a triangle, so by the 3-colorability it can be retracted to this triangle. By Lemma 2.1 0), H^* is a core so H_3 is in fact a triangle. But then the H^*-colouring is just a 3-colouring of graphs, a well-known NP-complete problem, contradicting the assumption that it is not NP-complete. □

3. Examples

In this section, we will consider some examples, that show that some of the steps in our proof are necessary. We will give examples of graphs which are do not have a 3 colouring, but which satisfy properties (1)-(4) from Lemma 2.1.

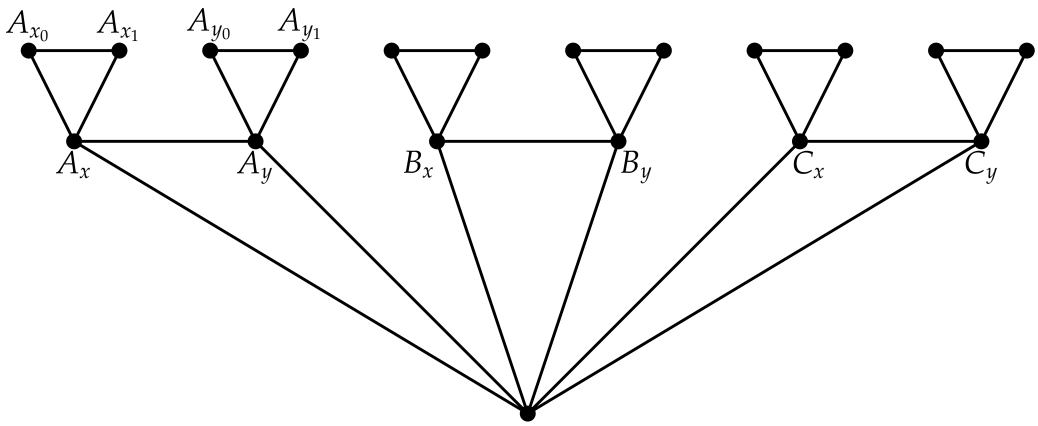
Example 3.1. *Let us consider a graph which has one vertex with five petals i.e. in the second level has ten vertices. Let us mark all vertices in the second level with 0 or 1, that is all petals have one vertex labelled with 0 and one vertex labelled with 1. In the third level this graph contains 16 vertices. The third level vertices will be connected to 5 of the vertices from the second level, one from each of the petals. Let us mark vertices from the third level with a five-digit binary number, with each digit corresponding to one of the five petals, and having the value of the label of the vertex on the corresponding petal with which it is connected. In our graph, we will only have vertices in the third level marked with even number of digits 1. In the third level two vertices are connected if their labels differ on four places. This graph does not have a 3 colouring, but satisfies properties (1)-(4) from Lemma 2.1.*

Note that the graph stays the same, if we flip the labelling on even number of petals (and pass the change to the markings on the third level accordingly). If we only flip labelling on a single petal, the graph follows the same rules, but has odd number of ones in the third level. Suppose we have a 3 colouring of this graph, and the central vertex is coloured by color number 2, while the other two colors are labelled 1 and 0. It is then easy to see that we can choose the labelling on the level 2 and these colours, so that colours correspond to the labelling, and the graph is still described as above. But then, each vertex from the third level, except for the vertex marked by 00000, has to be coloured with color 2 if there is a 3 colouring of this graph. However, since there are such vertices on the third level which are connected, we conclude that there can not be a 3 colouring.

It is also easy to see that this graph satisfies properties (1)-(4) from Lemma 2.1. Since evidently there is a triangle, we only need to check that every two vertices are connected with a path of length two, and that there is no K_4^- as a sub-graph. The non-trivial case is of two vertices from the third level or a vertex from a third and a vertex from a second level having a common neighbour. If a vertex A from the third level is not connected to a vertex B in the second level, it is connected to the vertex in the same petal, having the opposite label, and hence A and B have a common neighbour. Otherwise, if A is connected to B, then there is a vertex C in the third level that is connected to B but its marking is opposite to that of A for all other petals. But then C is connected to A and is common neighbour of A and B. Finally, consider two vertices A and B from the third level. Since number of ones in the markings of A and B are of the same parity, and the total number of petals is 5, i.e. odd, there must be a petal on which A and B have the same corresponding digit of the marking, i.e. they are both connected to the same vertex C from the second level.

To see that our graph does not contain K_4^- as a sub-graph, i.e. that no edge belongs to more than one triangle. Note that there is no triangle with all three vertices belonging to the third level: the markings of the two connected vertices from the third level differ on all but one places, so a path of length 3 would have at least two places on which three flips occur, and hence cannot be a closed path. It is easy to see that an edge with none of the vertices from the third level belongs to at most one triangle. If an edge has two vertices from the third level, then it is also clear that there can be only one vertex from level two with which both are connected. Similarly, if we have an edge connecting vertex A from level two, with vertex B from level three, it is clear that the unique vertex which is a common neighbour to A and B is the vertex from the third level, with the marking having the same digit as B corresponding to petal containing A, and opposite on all other places.

Example 3.2. *Let us consider a graph which has one vertex with three petals i.e. in the second level has six vertices. Let us denote the petals with A, B and C for convenience. Again, label the vertices on each petal with x and y, so that the second level contains vertices Ax, Ay, Bx, By, Cx and Cy. Now each of these vertices is connected with exactly two vertices from the third level, labelled with* 0 *and* 1*, in the form of disjoint, third level petals, i.e. so that the vertex from level two and the two vertices from level three a triangle. Besides the edges belonging to such triangles we will connect vertices from the level 3, according to the following rules. Each vertex from the third level based in A is connected to exactly one of the third level vertices from each of the third level petals based in B or C, and similarly for vertices based in B or C. In this way, two third level vertices from one third level petal can be connected to two vertices from the other third level petal, either so that the same labelled vertices are connected, or so that the opposite labelled vertices are connected. In the later case, we will say that there is a twist, otherwise we will say that there is no twist. Now, we join vertices based in A to vertices based in B without twist, the vertices based in B with vertices based in C without twist if and only if they have the same secondary label x or y, and the vertices based on A with vertices based in C with twist only if the secondary label at A is x. Again, this graph does not have a 3 colouring, but satisfies properties (1)-(4) from Lemma 2.1.*



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