Novel computational tool for coupling water and heat transport models – application on green roofs

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ABSTRACT

Introduction: Green roofs are one of the most common multifunctional types of Nature Based Systems (NBS) serving primarily for mitigation of the urban runoff (Stovin et al. 2012, Versini et al. 2020). Since relying on the soil water interaction, green roofs also have a significant impact on reduction of the local temperature, which has not been so deterministically investigated in the past. To simulate the change of substrate temperature and water content accurately and continuously, it is necessary to couple models for water and heat transport through (un)saturated porous media which has been done in many studies (Campbell 1985, Bittelli et al. 2008). The core of these models are the partial differential equations that are strongly nonlinear, especially Richards (1931) equation describing the unsaturated water flow, and hence their numerical solving is still challenging from the perspective of the computational time, numerical stability, and accuracy. Linearization of Richards equation has first been proposed by Ross (2003) who developed a stable explicit numerical scheme for solving it by using Taylor series and Kirchhoff potential to express unsaturated water fluxes, while similar approach has not been applied yet to Heat equation. The main deficiency of this approach as far as Richards equation is concerned is the necessity to use finer time discretization to avoid greater water balance errors, as well as the complex and often inaccurate transition from the unsaturated to saturated state and vice versa.

To develop a robust and accurate numerical tool for consecutive solving of Richards and Heat equations, several improvements compared to the existing approaches have been made. Firstly, Taylor series has also been applied on soil heat fluxes creating rather simple and mathematically elegant explicit numerical scheme for solving Heat equation. Secondly, unlike in Ross's method where only the first term of Taylor series is used, here are used the first and the second term to create more accurate approximation of water fluxes. Also, unlike in Ross (2003), here Richards equation is solved strictly with respect to Kirchhoff potential to smooth the transition between unsaturated and saturated water flow. Finally, the evapotranspiration rate at the top surface is not predefined but determined from the latent heat flux computed through the iterative solving of Richards and Heat equations. Here are presented preliminary simulation results of the proposed coupled model obtained by using approximately six days long timeseries of the measured meteorological data taken from Bittelli et al. (2008).

Methodology: Richards (1931) equation is the nonlinear partial differential equation of the second order describing the (un)saturated water flow through the porous media. If water fluxes q_w [m/s] are expressed through Taylor series using Kirchhoff potential $\phi(h_k) = \int_{-\infty}^{h_k} K_w(h_k)\,dh_k$, where $K_w(h_k)$ is the soil hydraulic conductivity [m/s] described with Brooks & Corey (1964) power law relation and h_k is the soil matric potential [m], Richards equation takes the following discrete form:

$$\left(\frac{\partial q_{w,i-1}}{\partial \phi_{i-1}}\bigg|^{k} + 0.5\,\frac{\partial^2 q_{w,i-1}}{\partial {\phi_{i-1}}^2}\bigg|^{k} \Delta\phi_{i-1}^{k+1,p-1}\right)\Delta\phi_{i-1}^{k+1,p} + \left(\frac{\partial q_{w,i-1}}{\partial \phi_{i}}\bigg|^{k} + 0.5\,\frac{\partial^2 q_{w,i-1}}{\partial {\phi_{i}}^2}\bigg|^{k} \Delta\phi_{i}^{k+1,p-1} - \frac{\partial q_{w,i}}{\partial \phi_{i}}\bigg|^{k} - 0.5\,\frac{\partial^2 q_{w,i}}{\partial {\phi_{i}}^2}\bigg|^{k} \Delta\phi_{i}^{k+1,p-1} - \frac{1}{\sigma}\frac{\Delta z_i}{\Delta t}\frac{1}{D_i^{k+1,p-1}}\right)\Delta\phi_{i}^{k+1,p} - \left(\frac{\partial q_{w,i}}{\partial \phi_{i+1}}\bigg|^{k} + 0.5\,\frac{\partial^2 q_{w,i}}{\partial {\phi_{i+1}}^2}\bigg|^{k} \Delta\phi_{i+1}^{k+1,p-1}\right)\Delta\phi_{i+1}^{k+1,p} = \frac{1}{\sigma}\left(q_{w,i}^{k} - q_{w,i-1}^{k} + \frac{\Delta z_i}{\Delta t}\left(\Delta\theta_i^{k+1,p-1} - \frac{\Delta\phi_i^{k+1,p-1}}{D_i^{k+1,p-1}}\right)\right)$$

(1)

where $D = \frac{d\phi}{d\theta}$ is diffusivity coefficient [m^2/s], and $\sigma = 0.5$ is the temporal weighting factor for unsaturated media (equal to 1.0 in case of saturation). Since using the second derivative $\frac{\partial^2 q}{\partial \phi^2}$ in Taylor series and *D* which is a function of ϕ, a system of *N* nonlinear equations (1) is formed, where *N* is the number of computational nodes along the soil depth. The system is solved iteratively (*p* indicates iteration number), and as a result $\Delta\phi_i^{k+1}$ values are obtained so that $\phi_i^{k+1} = \phi_i^k + \Delta\phi_i^{k+1}$ is determined for each node *i* = 1 to *N* at the following time step (*k+1*).

Heat equation is of the similar form as Richards equation, and hence it can be also solved through Taylor series by using solely the first derivative of the heat flux $q_{h,i} = \frac{K_{h,i}}{\Delta z_i}(T_i\text{-}T_{i+1})$ with respect to soil temperature *T*.

$$\left.\frac{\partial q_{h,i-1}}{\partial T_{i-1}}\right|_k \Delta T_{i-1}^{k+1} + \left(\left.\frac{\partial q_{h,i-1}}{\partial T_i}\right|_k - \left.\frac{\partial q_{h,i}}{\partial T_i}\right|_k - C_{h,i}\frac{\Delta z_i}{\sigma \Delta t}\right)\Delta T_i^{k+1} - \left.\frac{\partial q_{h,i}}{\partial T_{i+1}}\right|_k \Delta T_{i+1}^{k+1} = \frac{1}{\sigma}\left(q_{h,i}^k - q_{h,i-1}^k\right) \tag{2}$$

where C_h and K_h are the soil heat capacity [J/(m^3 K)] and the soil heat conductivity [W/(m K)], respectively, both depending solely on soil water content θ [-], while derivatives $\frac{\partial q_h}{\partial T}$ also have the analytical form as in case of Richards equation. If Equation (2) is written for each node *i* = 2 to *N-1* with exception of *i* = 1 and *N* where boundary conditions are defined, system of *N* linear equations is solved without iterations and $T_i^{k+1} = T_i^k + \Delta T_i^{k+1}$ values are obtained.

For solving both Richards and Heat equations, it is necessary to define top and bottom boundary conditions. In case of water transport, bottom boundary condition is set as free drainage ($q_{w,N} = K_{w,N}$), while at the top surface precipitation or irrigation rate is defined as a model input, as well as the evapotranspiration rate calculated from the results of the Heat equation. Heat flux at the bottom surface is computed based on the defined temperature of a roof deck $q_{h,N} = \frac{K_{h,N}}{\Delta z_N}(T_N\text{-}T_{bott})$, while the difference $R_n - \lambda ET - H$, where R_n is the known net radiation [W/m^2] while λET and *H* are the latent and the sensible heat fluxes [W/m^2], respectively, is used at the top surface:

$$\lambda ET = \frac{\rho_a c_p}{\gamma}\frac{(e_s(T_1)\text{-}e_a)}{r_s + r_a}\frac{H_r(T_1)\text{-}H_a}{1\text{-}H_a} \tag{3}$$

$$H = \frac{\rho_a c_p}{r_a}(T_1\text{-}T_{air}) \tag{4}$$

where ρ_a is the air density [kg/m^3] depending on the air temperature T_{air} [K] and atmospheric pressure, c_p = 1013 J/(K kg) is the specific heat of air, γ is the psychometric constant [Pa/K] depending on the atmospheric pressure, $\lambda = 2.45$ MJ/kg is the latent heat of vaporization, r_s and r_a are the soil surface and aerodynamic resistances for water vapour transfer [s/m], e_s and e_a are the vapour pressures [Pa] at the soil surface and in the air, respectively, while H_a and H_r are the relative air humidity [-] and the relative humidity at the soil surface [-], respectively.

The calculation is performed by initially solving the Heat equation with $R_n - \lambda ET - H$ used at the top boundary, where λET and *H* are expressed by means of Taylor series using the values T_i^k from the previous time step. After computing values of T_i^{k+1}, new value of λET is determined (Eq. 3) and transformed into the evapotranspiration rate ET = $\lambda ET/\lambda$ to be used as an input for Richards equation. After solving system of *N* nonlinear equations (1), ϕ_i^{k+1} and hence θ_i^{k+1} values are obtained and used again to recalculate the soil thermal properties and determine new T_i^{k+1} values. The whole procedure is iteratively repeated until the difference between λET values in two consecutive iterations becomes negligible.

Preliminary results: Figure 1 illustrates how does the soil water content affects the substrate temperature (middle graph), and the latent heat flux (bottom graph) which has the same character as the evapotranspiration rate (see Eq. 3). With solid lines is presented initially saturated substrate that is not irrigated during the simulation period, while dashed lines represent initially saturated substrate that is irrigated during six hours each day. Calculation is performed for a 20 cm thick low permeable green roof substrate (saturated hydraulic conductivity 3x10^{-6} m/s) with porosity 0.4. Results show strong dependence between the soil water content and both the latent heat flux and soil temperature. When the substrate is wet, the energy coming from the net radiation is mostly used for vaporization, but as soon as the substrate becomes dry this energy is mostly used for heating up the substrate. In Figure 1 this becomes evident after 90 h from the beginning of simulation, when the latent heat flux for non-irrigated substrate (solid lines in Figure 1) starts decreasing due to the lack of water, while at the same time substrate temperature starts rising significantly. On the other hand, if green roof is irrigated frequently enough (dashed lines in Figure 1) so to not allow the water content to decrease significantly (approaching the residual water content), the latent heat flux becomes larger and the substrate temperature decreases.

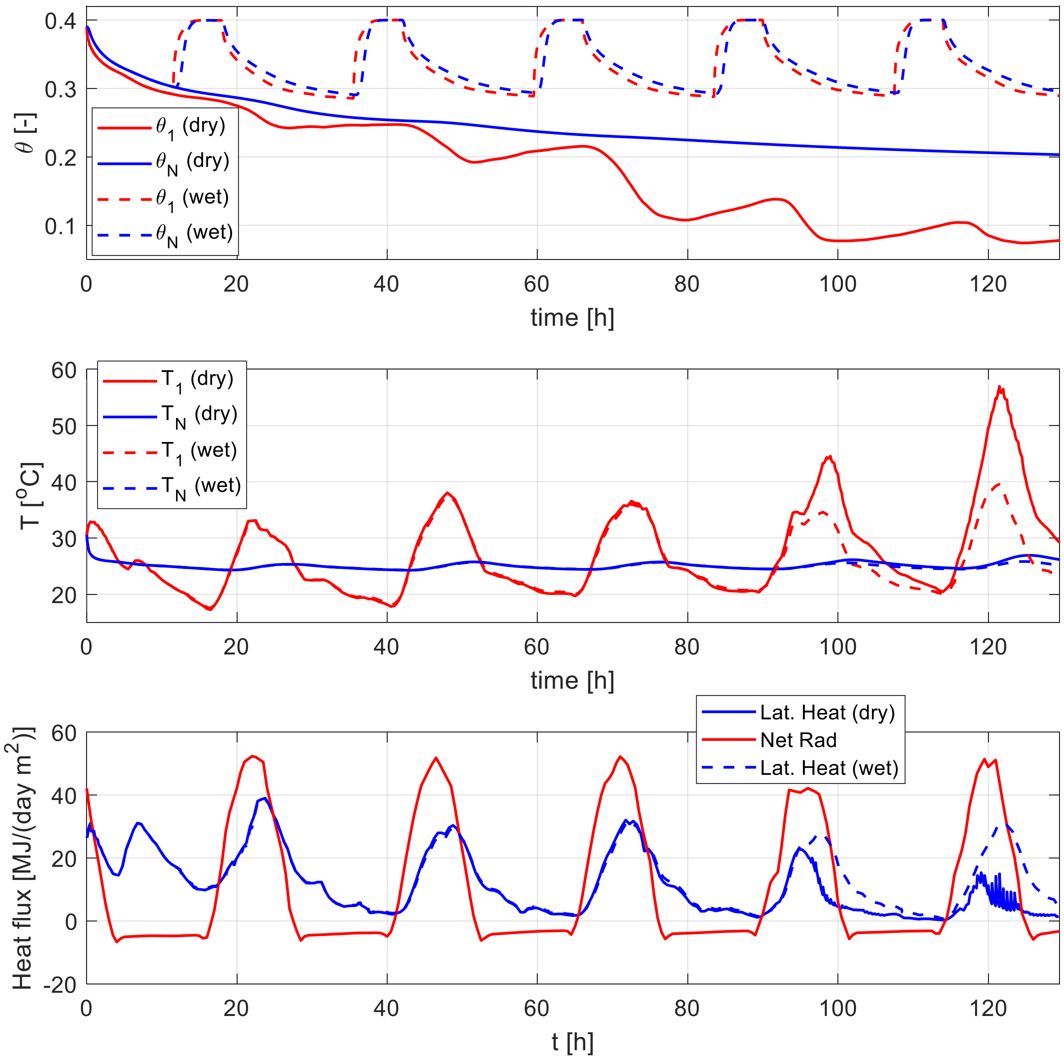

Figure 1. Change of substrate water content (top graph), substrate temperature (middle graph) and Latent Heat Flux (bottom graph) depending on whether the green roof is irrigated (dashed lines) or not (solid lines)

Due to utilization of the first and the second term of Taylor series when solving Richards equation, the calculation is more accurate but also more efficient since larger time steps are used. For the example presented here, the total water balance error (water volume per specific area) is order of magnitude of 10^{-8} m^3/m^2, while the total energy balance error is about 10^{-10} J/m^2.

Conclusion: The proposed coupled water and heat transport model relies on iterative solving of Richards equation and Heat equation with variable boundary conditions. It relies on the linearization of highly nonlinear partial differential equations by using Taylor series to express water and heat fluxes and create that way more accurate, robust, and stable numerical schemes widely applicable. Besides the numerical schemes, an innovation brought by this work is the computation of the evapotranspiration rate through the iterative determination of various energy balance components. Simulation results presented here are logical, indicating the importance of irrigation of green roofs and other similar NBS so they do not only serve as a detention element for stormwater, to mitigate floods, but also to reduce the temperature of the soil and hence to mitigate the urban heat island effect pronounced during droughts.

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