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ON THE EXTENSION OF THE ERDÖS–MORDELL TYPE INEQUALITIES

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Abstract. We discuss the extension of inequality $R_A \geq \frac{c}{a} r_b + \frac{b}{a} r_c$ to the plane of triangle $\triangle ABC$. Based on the obtained extension, in regard to all three vertices of the triangle, we get the extension of Erdös-Mordell inequality, and some inequalities of Erdös-Mordell type.

1. Introduction

Let triangle $\triangle ABC$ be given in Euclidean plane. Denote by R_A, R_B and R_C the distances from the arbitrary point M in the interior of $\triangle ABC$ to the vertices A, B and C respectively, and denote by r_a, r_b and r_c the distances from the point M to the sides BC, CA and AB respectively (Figure 1).

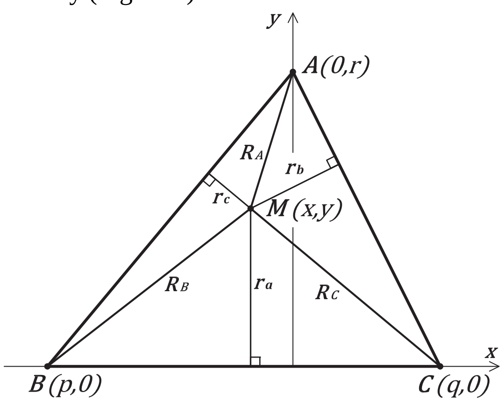

Figure 1: Erdös-Mordell inequality

Then Erdös-Mordell inequality is true:

$$R_A + R_B + R_C \geq 2\,(r_a + r_b + r_c) \qquad (1)$$

whereat equality holds if and only if triangle ABC is equilateral and M is its center. This inequality was conjectured by P. Erdös as Amer. Math. Monthly Problem 3740 in 1935. [9], after his experimental conjecture in 1932. [13]. It was proved by L.J. Mordell in 1935. (in Hungarian, according to [13]), and as the solution of the Problem 3740 in 1937. [22].

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Considering the Erdös-Mordell inequality (1) the goal of this research is to determine areas in the plane of the triangle, where the following three inequalities are valid:

$$R_A \geq \frac{c}{a} r_b + \frac{b}{a} r_c \tag{2}$$

$$R_B \geq \frac{c}{b} r_a + \frac{a}{b} r_c \tag{3}$$

$$R_C \geq \frac{b}{c} r_a + \frac{a}{c} r_b \tag{4}$$

where $a = |BC|$, $b = |CA|$, $c = |AB|$.

In this paper we determine a set of points $\boldsymbol{E}$ for which

$$R_A + R_B + R_C \geq \left(\frac{c}{b} + \frac{b}{c}\right) r_a + \left(\frac{c}{a} + \frac{a}{c}\right) r_b + \left(\frac{a}{b} + \frac{b}{a}\right) r_c \tag{5}$$

is valid. It is known that the triangular area of $\triangle ABC$ is contained in the set $\boldsymbol{E}$ [3], [4], [11], [13], [14], [26]. Here we show that the set $\boldsymbol{E}$ is greater than the triangle $\triangle ABC$, and we give a geometric interpretation of the set $\boldsymbol{E}$.

The proofs of Erdös-Mordell inequality are often based on different proofs of inequality (2), as given in [4], [6], [7], [11], [12], [23], [26]. N. Derigades in [8] proved the inequality (5) valid in the whole plane of the triangle, where r_a, r_b and r_c, are signed distances. A similar result was given by B. Malešević [20], [21].

Note that V. Pambuccian [24] recently proved that the Erdös-Mordell inequality is equivalent to non-positive curvature. Overview of recent results on Erdös-Mordell inequalities and related inequalities is given in [1] - [3], [5], [8], [10], [13] - [21], [24], [25], [27] - [30] .

2. The Main Results

In this section we analyze only the inequality (2). Let $\triangle ABC$ be a triangle with vertices $A(0,r), B(p,0), C(q,0)$, $p \neq q, r \neq 0$. Without diminishing generality, let $p < q$. We denote by $M(x,y)$ an arbitrary point in the plane of the triangle $\triangle ABC$. The distance from the point M to the point A, and the distance from the point M to the straight lines $\boldsymbol{b}$ and $\boldsymbol{c}$ are given by functions:

$$R_A = \sqrt{x^2 + (y-r)^2} \tag{6}$$

$$r_b = \frac{|-qy - rx + qr|}{\sqrt{r^2 + q^2}} \tag{7}$$

$$r_c = \frac{|py + rx - pr|}{\sqrt{r^2 + p^2}} \tag{8}$$

respectively. Consider the inequality (2) related to the vertex A. The analytical notation of this inequality is:

$$\sqrt{x^2 + (y-r)^2} \geq \frac{\sqrt{r^2+p^2}}{|q-p|} \frac{|-qy - rx + qr|}{\sqrt{r^2+q^2}} + \frac{\sqrt{r^2+q^2}}{|q-p|} \frac{|py + rx - pr|}{\sqrt{r^2+p^2}}, \tag{9}$$

i.e.

$$|q-p|\sqrt{r^2+p^2}\sqrt{r^2+q^2}\sqrt{x^2+(y-r)^2} \;\geq\; \left(r^2+p^2\right)|-qy-rx+qr| \\ +\left(r^2+q^2\right)|py+rx-pr|. \tag{10}$$

Let $y=kx+r, k\in\overline{\mathbb{R}}$, then the inequality (10) reads as follows:

$$|x|\,|q-p|\sqrt{r^2+p^2}\sqrt{r^2+q^2}\sqrt{1+k^2}\geq|x|\left(\left(r^2+p^2\right)|-qk-r|+\left(r^2+q^2\right)|pk+r|\right) \tag{11}$$

For $x=0$, the previous inequality is reduced to an equality which solution is the point $A(0,r)$. For $x\neq 0$ we obtain inequality by a single variable k:

$$|q-p|\sqrt{r^2+p^2}\sqrt{r^2+q^2}\sqrt{1+k^2}\geq\left(r^2+p^2\right)|-qk-r|+\left(r^2+q^2\right)|pk+r|. \tag{12}$$

Solution of the inequality (12) reduces to four cases per parameter k:

$$(\alpha_1):\begin{cases} pk+r\geq 0\\ -qk-r\geq 0,\end{cases} \tag{13}$$

$$(\alpha_2):\begin{cases} pk+r< 0\\ -qk-r\geq 0,\end{cases} \tag{14}$$

$$(\alpha_3):\begin{cases} pk+r\geq 0\\ -qk-r< 0,\end{cases} \tag{15}$$

$$(\alpha_4):\begin{cases} pk+r< 0\\ -qk-r< 0.\end{cases} \tag{16}$$

Note that the value k corresponds to the points $(x,y)\in\mathbb{R}^2$ located on the straight line $y=kx+r$. With its values, the mentioned parameter of the line $y=kx+r$ decomposes $\mathbb{R}^2$ on four corner areas. Inquiring the existence of parameter k (i.e. the pencil of lines $y=kx+r$ through the vertex A) depending on the signs of parameters p,q and r, we provide the following table of existing corner areas $(\alpha_1)-(\alpha_4)$:

	p	q	r	(α_1)	(α_2)	(α_3)	(α_4)
1.	>0	>0	>0	+	+	+	−
2.	<0	>0	>0	+	−	+	+
3.	<0	<0	>0	−	+	+	+
4.	>0	>0	<0	−	+	+	+
5.	<0	>0	<0	+	+	−	+
6.	<0	<0	<0	+	+	+	−
7.	$=0$	>0	>0	+	−	+	−
8.	$=0$	>0	<0	−	+	−	+
9.	<0	$=0$	>0	−	−	+	+
10.	<0	$=0$	<0	+	+	−	−

Table 1: The existence of the corner area depending on the parameters p, q and r

The corner areas (α_1) and (α_4) are always in the interior of $\sphericalangle BAC$ and its cross angle, while the areas (α_2) and (α_3) are in the interior of its supplementary angle (Figure 2).

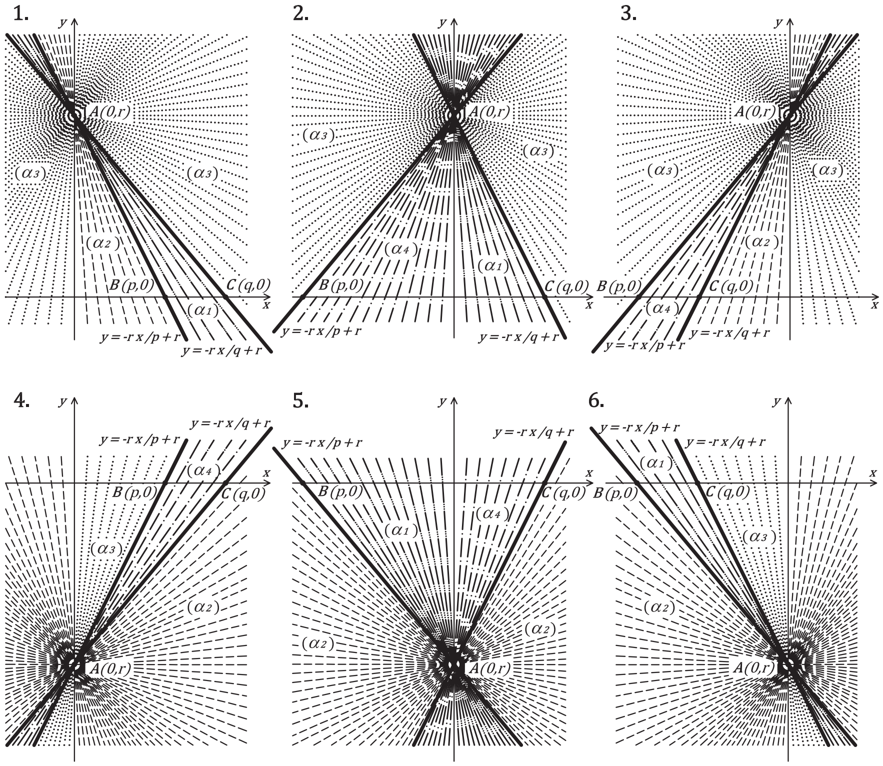

Figure 2: Existence of the corner area for the vertex A (Cases 1. to 6. in the Table 1)

Let us consider the equation:

$$(q-p)\sqrt{r^2+p^2}\sqrt{r^2+q^2}\sqrt{1+k^2} = \left(r^2+p^2\right)|-qk-r| + \left(r^2+q^2\right)|pk+r|. \tag{17}$$

I) Let k fulfill (α_1) or (α_4). Then the previous equation can be rewritten in a way that follows, with positive sign (+) in the case of area (α_1) and negative sign (−) in the case of area (α_4)

$$(q-p)\sqrt{r^2+p^2}\sqrt{r^2+q^2}\sqrt{1+k^2} = \pm\left((-qk-r)\left(r^2+p^2\right) + (pk+r)\left(r^2+q^2\right)\right) \tag{18}$$

i.e.

$$(q-p)\sqrt{r^2+p^2}\sqrt{r^2+q^2}\sqrt{1+k^2} = \pm(q-p)\left(r(q+p) + k\left(pq-r^2\right)\right) \tag{19}$$

abbreviated written as

$$\lambda\sqrt{1+k^2} = \pm\beta k \pm \gamma = \begin{cases} \beta k+\gamma, & k\in(\alpha_1) \\ -\beta k-\gamma, & k\in(\alpha_4) \end{cases} \tag{20}$$

where at:

$$\lambda = (q-p)\sqrt{r^2+p^2}\sqrt{r^2+q^2} \quad \text{and} \quad \lambda > 0 \tag{21}$$

$$\beta = \left(pq-r^2\right)(q-p) \tag{22}$$

$$\gamma = r\left(q^2-p^2\right). \tag{23}$$

As $p<q$, the equation (19) can be divided by $q-p\neq 0$ and then squared:

$$\left(r^2+p^2\right)\left(r^2+q^2\right)\left(1+k^2\right) = \left(r(q+p)+k\left(pq-r^2\right)\right)^2 \tag{24}$$

which transforms into

$$\left(r(p+q)k-\left(pq-r^2\right)\right)^2 = 0. \tag{25}$$

Based on the above equation, we conclude that there exists the unique solution:

$$k_1 = \frac{pq-r^2}{r(p+q)} \tag{26}$$

only if, for $k=k_1$:

$$\pm\beta k \pm \gamma \geq 0 \tag{27}$$

is valid.

Hence, the straight line $y=k_1x+r$ is in the interior of $\sphericalangle BAC$ and its cross angle, or it doesn't exist. The cases where values k_1 from the formula (26) does not meet the condition (27) are presented in the *Table 1* with:

in the case 1: $k_1 > -r/q \Longleftrightarrow p\left(q^2+r^2\right) > 0$;

in the case 3: $k_1 > -r/p \Longleftrightarrow (-q)\left(p^2+r^2\right) > 0$;

in the case 4: $k_1 < -r/q \Longleftrightarrow p\left(q^2+r^2\right) > 0$;

in the case 6: $k_1 < -r/p \Longleftrightarrow (-q)\left(p^2+r^2\right) > 0$.

LEMMA 1. *For* $k\in(\alpha_1)\cup(\alpha_4)$ *inequality* (12) *is valid, where equality holds for* $k=k_1$ *if* (27) *is fulfilled.*

Proof. (12) $\Longleftrightarrow \left(r(p+q)k-\left(pq-r^2\right)\right)^2 \geq 0$. □

COROLLARY 1. *Inequality* (12) *is valid for lines* ***b*** *and* ***c***.

II) Let k fulfill (α_2) or (α_3). Then equation (17) can be rewritten in a way that follows, with negative sign (−) in the case of area (α_2) and positive sign (+) in the case of area (α_3)

$$(q-p)\sqrt{r^2+p^2}\sqrt{r^2+q^2}\sqrt{1+k^2} = \pm\left((qk+r)\left(r^2+p^2\right)+(pk+r)\left(r^2+q^2\right)\right) \tag{28}$$

or abbreviated written as

$$\lambda\sqrt{1+k^2}=\pm\delta k\pm\varepsilon=\begin{cases}\ \ \delta k+\varepsilon, & k\in(\alpha_3)\\ -\delta k-\varepsilon, & k\in(\alpha_2)\end{cases} \tag{29}$$

with parameters:

$$\lambda=(q-p)\sqrt{r^2+p^2}\sqrt{r^2+q^2} \quad \text{and} \quad \lambda>0$$

$$\delta=\left(r^2+pq\right)(q+p) \tag{30}$$

$$\varepsilon=r\left(2r^2+q^2+p^2\right). \tag{31}$$

The equation (29) is considered under the following condition:

$$\pm\,\delta k\pm\varepsilon\geq 0. \tag{32}$$

By squaring the equation (29) we obtain

$$P(k)=\lambda^2\left(1+k^2\right)-(\pm\delta k\pm\varepsilon)^2=\left(\lambda^2-\delta^2\right)k^2-2\delta\varepsilon k+\left(\lambda^2-\varepsilon^2\right)=0. \tag{33}$$

For the square trinomial

$$P(k)=\widehat{\mathrm{A}}\,k^2+\widehat{\mathrm{B}}\,k+\widehat{\mathrm{C}} \tag{34}$$

coefficients $\widehat{\mathrm{A}}$, $\widehat{\mathrm{B}}$, $\widehat{\mathrm{C}}$ are determined by:

$$\widehat{\mathrm{A}}=\lambda^2-\delta^2=(q-p)^2\left(r^2+p^2\right)\left(r^2+q^2\right)-\left(r^2+pq\right)^2(q+p)^2 \tag{35}$$

$$\widehat{\mathrm{B}}=-2\delta\varepsilon=-2r\left(r^2+pq\right)(q+p)\left(2r^2+q^2+p^2\right) \tag{36}$$

$$\widehat{\mathrm{C}}=\lambda^2-\varepsilon^2=\left(r^2+pq\right)\left(\left(pq-r^2\right)(q-p)^2-2r^2(2r^2+q^2+p^2)\right). \tag{37}$$

Let us consider the equation:

$$\widehat{\mathrm{A}}=-4pqr^4+\left(p^4+q^4-4pq^3-4p^3q-2p^2q^2\right)r^2-4p^3q^3=0. \tag{38}$$

It has real solutions for r in the following form:

$$\begin{cases} r_{1,2}=\dfrac{1}{4\sqrt{pq}}\left((q-p)^2\ \pm\ \sqrt{(q-p)^4-16p^2q^2}\ \right)>0\\ r_{3,4}=-\dfrac{1}{4\sqrt{pq}}\left((q-p)^2\ \pm\ \sqrt{(q-p)^4-16p^2q^2}\ \right)<0\end{cases} \tag{39}$$

iff

$$\left(p\geq 0\ \wedge\ q\geq(3+2\sqrt{2})p\right)\ \vee\ \left(p<0\ \wedge\ q\leq(3-2\sqrt{2})p\right). \tag{40}$$

REMARK 1. When $p<0$ and $q>0$ then $\widehat{\mathrm{A}}=4\,|p|\,qr^4+\left(q^2-p^2\right)^2r^2+4\,|p|\,q\left(p^2+q^2\right)r^2+4\,|p|^3q^3>0$ is valid. Note that the equation $\widehat{\mathrm{A}}=0$ is not considered for $p=0$ or $q=0$ (because we obtain the contradictions: $p=0,\ q\neq 0$: $\widehat{\mathrm{A}}=r^2q^4=0\Longrightarrow r=0$; i.e. $p\neq 0,\ q=0$: $\widehat{\mathrm{A}}=r^2p^4=0\Longrightarrow r=0$).

We distinguish the cases:

a) Let $r = r_j$ for some $j = 1, 2, 3, 4$, then $\widehat{\mathrm{A}} = 0$. In this case, $\widehat{\mathrm{B}} \neq 0$, because $r^2 + pq \neq 0$ and $q + p \neq 0$ $\big($in the case of equilateral triangle, there will be valid $q + p = 0$ and then $r = \pm pi,\ i = \sqrt{-1}\,\big)$. Therefore, by solving the linear equation $\widehat{\mathrm{B}}\,k + \widehat{\mathrm{C}} = 0$ we find that:

$$k_2 = -\frac{\widehat{\mathrm{C}}}{\widehat{\mathrm{B}}} = \frac{\lambda^2 - \varepsilon^2}{2\delta\varepsilon} = \frac{(q-p)^2\left(r^2+p^2\right)\left(r^2+q^2\right) - r^2\left(2r^2+q^2+p^2\right)^2}{2r\,(q+p)\,(2r^2+q^2+p^2)}. \tag{41}$$

For $p < 0$ and $q > 0$ the case **a)** is not considered $\big($because $\widehat{\mathrm{A}} > 0\big)$. Let us examine when the value k_2 meet the condition (32). It is valid that:

$$\pm\delta k_2 \pm \varepsilon \geq 0 \iff \pm(\delta k_2 + \varepsilon) = \pm\left(\delta\frac{\lambda^2-\varepsilon^2}{2\delta\varepsilon} + \varepsilon\right) = \pm\left(\frac{\lambda^2+\varepsilon^2}{2\varepsilon}\right) \geq 0.$$

Based on $\varepsilon = r\left(2r^2 + q^2 + p^2\right)$ we conclude:
if $r > 0$ then $\delta k_2 + \varepsilon \geq 0$ is fulfilled, whereby k_2 fulfills condition (32) and $k_2 \in (\alpha_3)$;
if $r < 0$ then $-\delta k_2 - \varepsilon \geq 0$ is fulfilled, whereby k_2 fulfills condition (32) and $k_2 \in (\alpha_2)$.
In this case, the line $y = k_2 x + r$ is in the exterior of $\sphericalangle BAC$ and its cross angle.

b) Let $r \neq r_j$ for each $j = 1, 2, 3, 4$, then $\widehat{\mathrm{A}} \neq 0$ and in this case, by solving the quadratic equation (33), we find the values:

$$\begin{aligned} k_{2,3} &= \frac{-\delta\varepsilon \pm \sqrt{\lambda^2\,(\delta^2+\varepsilon^2-\lambda^2)}}{\delta^2-\lambda^2} \\ &= \frac{r(p+q)(r^2+pq)(q^2+p^2+2r^2) \pm 2\left(r^2+p^2\right)\left(r^2+q^2\right)(q-p)\sqrt{r^2+pq}}{(q-p)^2\,(r^2+p^2)\,(r^2+q^2) - (r^2+pq)^2\,(q+p)^2}. \end{aligned} \tag{42}$$

If $r^2 + pq \geq 0$ then exists $k_{2,3} \in \mathbb{R}$. Incidence of $k_{2,3} \in \mathbb{R}$ to the area (α_3), as to the area (α_2) is determined by the inequality (32). The expression $\delta k_{2,3} + \varepsilon$ exists for $\delta \neq \pm\lambda$, whereby the expression $\delta k_{2,3} + \varepsilon$ is either positive or negative (because $\delta k_{2,3} + \varepsilon = 0 \Longrightarrow \delta = \pm\lambda$).
Based on the Corollary 1, the straight lines $y = k_s x + r, (s = 2,\ 3)$ are in the exterior of $\sphericalangle BAC$ and its cross angle (*Figure 3*).
Consider the limiting case for $k_{2,3}$ when $r \to r_j$. Note that $\widehat{\mathrm{A}} = \lambda^2 - \delta^2 \underset{r \to r_j}{\longrightarrow} 0$ is valid, whereat from

$$k_{2,3} = \frac{-\varepsilon}{(\delta-\lambda)\,(\delta+\lambda)} \cdot \left(\delta \mp |\lambda|\sqrt{1 + \frac{\delta^2-\lambda^2}{\varepsilon^2}}\right)$$

follows

$$\lim_{r \to r_j} k_2 = \frac{-\varepsilon}{(\delta+\lambda)} \quad \wedge \quad \lim_{r \to r_j} k_3 = \infty.$$

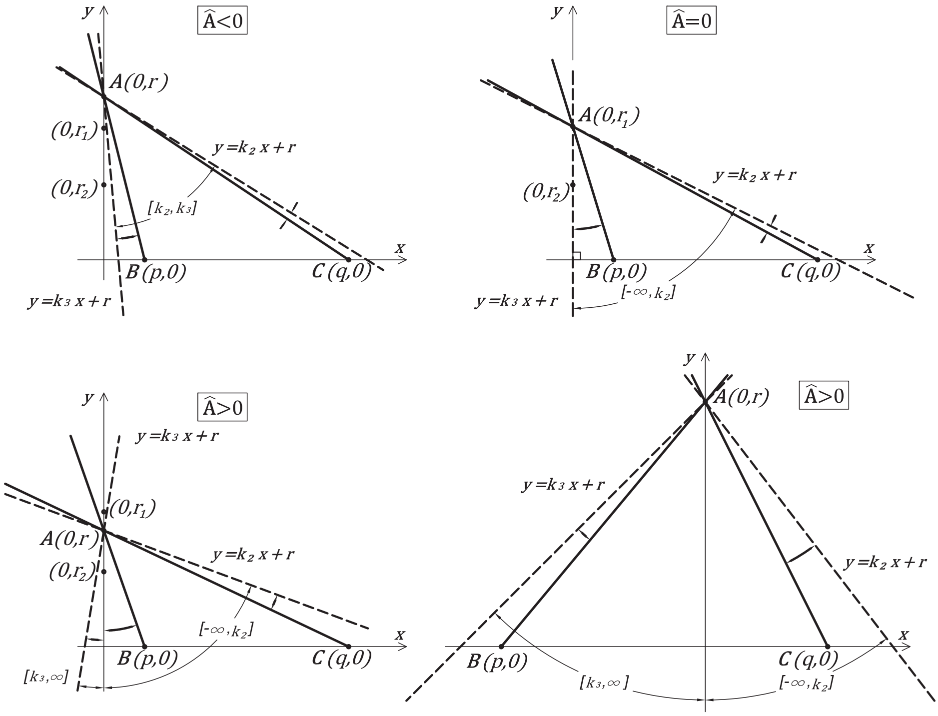

Figure 3: The existence of the lines $y = k_s x + r$, $(s = 2, 3)$
depending on the parameter $\widehat{\mathrm{A}}$

Related to the $\sphericalangle BAC$ we distinguish the cases:

1. $\sphericalangle BAC < \pi/2 \Longleftrightarrow r^2 + pq > 0$ and if $\widehat{\mathrm{A}} \neq 0$ then there are two real and different values of k_2 and k_3. In this case, the following lemma is valid:

LEMMA 2. *For* $\sphericalangle BAC < \pi/2$, $k \in (\alpha_2) \cup (\alpha_3)$ *the inequality* (12) *is valid, just in the cases:*

1. $\widehat{\mathrm{A}} > 0 \wedge k \in [-\infty, k_2] \cup [k_3, +\infty] \setminus \big((\alpha_1) \cup (\alpha_4)\big)$;
2. $\widehat{\mathrm{A}} = 0 \wedge k \in [-\infty, k_2] \setminus \big((\alpha_1) \cup (\alpha_4)\big)$;
3. $\widehat{\mathrm{A}} < 0 \wedge k \in [k_2, k_3] \setminus \big((\alpha_1) \cup (\alpha_4)\big)$;

where the equality holds for $k = k_2$ *or* $k = k_3$.

2. If $\sphericalangle BAC = \pi/2 \Longleftrightarrow r^2 + pq = 0$ then $\widehat{\mathrm{A}} = -qp\,(q-p)^4$, $\widehat{\mathrm{B}} = 0$ and $\widehat{\mathrm{C}} = 0$, according to the equation (42) that $k_{2,3} = 0$. Hence is valid:

LEMMA 3. *For* $\sphericalangle BAC = \pi/2$ *and* $k \in (\alpha_2) \cup (\alpha_3)$ *the inequality* (12) *is valid. The equality is valid only for* $k = 0$.

Proof. (12) $\Longleftrightarrow \widehat{\mathrm{A}}\,k^2 + \widehat{\mathrm{B}}\,k + \widehat{\mathrm{C}} \geq 0 \Longleftrightarrow -qp\,(q-p)^4 k^2 \geq 0$. □

3. $\sphericalangle BAC > \pi/2 \Longleftrightarrow r^2 + pq < 0$. In this case, for: $r^2 < -pq$ and for the coefficient $\widehat{\mathrm{A}}$:

$$\widehat{\mathrm{A}} > 4r^6 + \left(p^4 + q^4\right) r^2 + 4\left(p^2 + q^2\right) r^4 - 2r^6 + 4p^2q^2r^2$$
$$= 2r^6 + 4\left(p^2 + q^2\right) r^4 + \left(p^4 + q^4 + 4p^2q^2\right) r^2 > 0$$

is valid. Since $k_{2,3} \in \mathbb{C}$ and $\widehat{\mathrm{A}} > 0$ the inequality (12) is valid, which proves the claim:

LEMMA 4. *For $\sphericalangle BAC > \pi/2$ and $k \in (\alpha_2) \cup (\alpha_3)$ the inequality* (12) *is valid in the strict form.*

Based on the previous considerations in **I)** and **II)**, follows:

STATEMENT 1. *The inequality* (12) *holds in following cases:*

$$k \in (\alpha_1) \cup (\alpha_4)$$

or

$$k \in (\alpha_2) \cup (\alpha_3) \ \textit{ for } \ \sphericalangle BAC \geq \pi/2$$

i.e.

$$k \in [-\infty, k_2] \cup [k_3, +\infty] \setminus \big((\alpha_1) \cup (\alpha_4)\big) \ \wedge \ \widehat{\mathrm{A}} > 0$$
$$k \in [-\infty, k_2] \setminus \big((\alpha_1) \cup (\alpha_4)\big) \ \wedge \ \widehat{\mathrm{A}} = 0$$
$$k \in [k_2, k_3] \setminus \big((\alpha_1) \cup (\alpha_4)\big) \ \wedge \ \widehat{\mathrm{A}} < 0,$$

for $\sphericalangle BAC < \pi/2$.

3. Conclusion

For the vertex A, let us define

$$\boldsymbol{E}_A = \left\{(x,y) \,\middle|\, R_A \geq \frac{c}{a} r_b + \frac{b}{a} r_c \right\},$$

and for the vertices B and C, let us define

$$\boldsymbol{E}_B = \left\{(x,y) \,\middle|\, R_B \geq \frac{c}{b} r_a + \frac{a}{b} r_c \right\},$$
$$\boldsymbol{E}_C = \left\{(x,y) \,\middle|\, R_C \geq \frac{b}{c} r_a + \frac{a}{c} r_b \right\},$$

respectively. Based on the analysis of the inequalities (2), (3) and (4), the inequality (5) is valid in the intersection of the areas:

$$\boldsymbol{E} = \boldsymbol{E}_A \cap \boldsymbol{E}_B \cap \boldsymbol{E}_C. \tag{43}$$

Therefore follows

STATEMENT 2. *Erdös-Mordell inequality is valid in the area* $\boldsymbol{E}$.

Let us define the set $\boldsymbol{M}$ by the intersection of the corner areas formed from $\boldsymbol{E}_A$, $\boldsymbol{E}_B$ and $\boldsymbol{E}_C$, containing the initial triangle. Then the set of points $\boldsymbol{M}$ is quadrilateral or hexagonal shape, and is contained the area $\boldsymbol{E}$ (*Figure 4*).

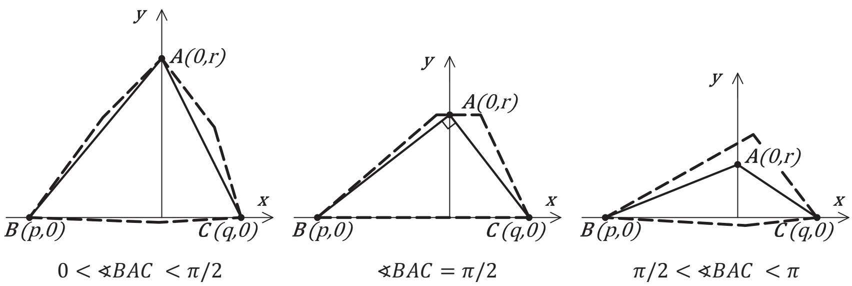

Figure 4: Extension of the triangle ABC to the area $\mathbf{M} \subset \mathbf{E}$

Let us define Erdös-Mordell curve in the plane of triangle, by the following equation:

$$R_A + R_B + R_C = 2\left(r_a + r_b + r_c\right), \tag{44}$$

where

$$R_A = \sqrt{x^2 + (y-r)^2}, \quad R_B = \sqrt{(x-p)^2 + y^2}, \quad R_C = \sqrt{(x-q)^2 + y^2},$$

$$r_a = \frac{|y(q-p)|}{\sqrt{(q-p)^2}} = |y|, \quad r_b = \frac{|-q(y-r) - rx|}{\sqrt{r^2+q^2}}, \quad r_c = \frac{|-p(y-r) - rx|}{\sqrt{r^2+p^2}}.$$

The curve (44) is a union of parts of algebraic curves of order eight (Figure 5).

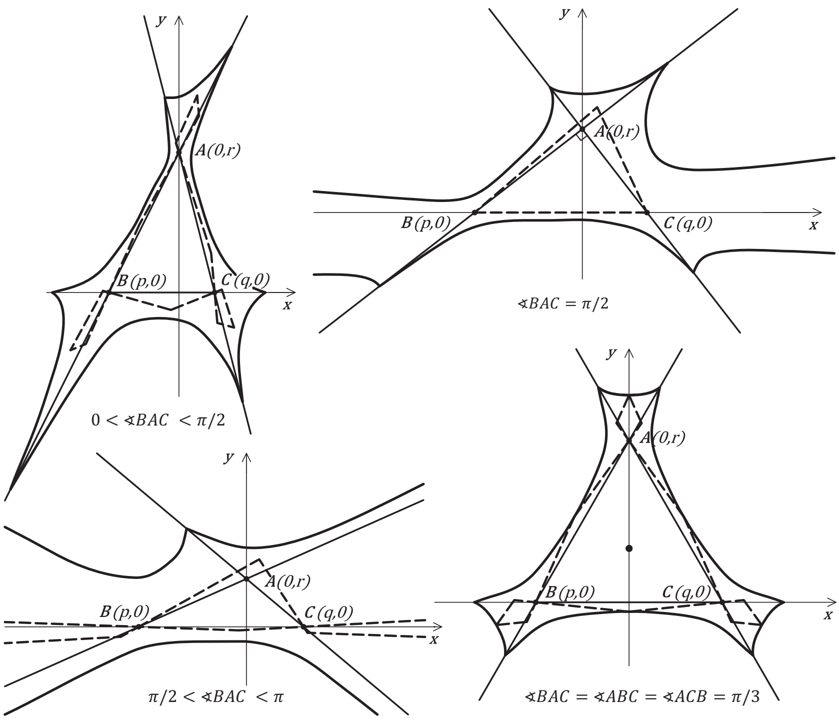

Figure 5: Erdös-Mordell curve and the area $\mathbf{E}$

Let us denote by ***E'*** the part of the plane $\mathbb{R}^2$ bounded by the Erdös-Mordell's curve and consisting the triangle $\triangle ABC$. Thus, according to the fact that inequality (5) is valid in the area of the triangle $\triangle ABC$, and based on continuity, it follows that inequality (5) is valid in the area ***E'***. Remark that the area ***E'*** allows us to precise when, except for the inequality (5), some of the inequalities (2), (3) and/or (4) are true. For example, in the area $(\boldsymbol{E'}\backslash\boldsymbol{E}_A)\cap\boldsymbol{E}_B\cap\boldsymbol{E}_C$ the inequalities (5), (4), (3) are true and (2) is not true. At end of this section let us emphasize that the following statement is true.

STATEMENT 3. *All geometric inequalities based on the inequalities* (2), (3) *and* (4) *can be extended from the triangle interior to the area* ***E*** .

EXAMPLE 1. In the area ***E***, the inequality of Child [7] is valid:

$$R_A \cdot R_B \cdot R_C \geq 8 \cdot r_a \cdot r_b \cdot r_c \tag{45}$$

because, based on inequality between arithmetic and geometric mean, follows:

$$a \cdot R_A \geq b \cdot r_c + c \cdot r_b \geq 2\sqrt{b \cdot c \cdot r_b \cdot r_c} \tag{46}$$

$$b \cdot R_B \geq c \cdot r_a + a \cdot r_c \geq 2\sqrt{c \cdot a \cdot r_c \cdot r_a} \tag{47}$$

$$c \cdot R_C \geq a \cdot r_b + b \cdot r_a \geq 2\sqrt{a \cdot b \cdot r_a \cdot r_b}. \tag{48}$$

Hence, by multiplying the left and right sides of inequalities (46) - (48), we get the inequality (45) in the area ***E***. □

At the end of this paper, let us set up an open problem (proposed by anonymous reviewer): prove or disprove that there exist a positive number ε such that the area of ***E'*** is bigger than $1+\varepsilon$ times the area of the triangle for every triangle. Thus, we set a conjecture: for the finite area of ***E'*** the value ε is determined in the case of equilateral triangle.

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