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Jočković, M., Nefovska-Danilović, M., Petronijević, M., 2015. Free vibration analysis of plate assemblies using dynamic stiffness method, in: *Stability, Vibration, and Control of Machines and Structures : Keynote and Invited Lectures of the International Symposium on Stability, Vibration, and Control of Machines and Structures* Belgrade, July 3-5,2014, Springer International Publishing, Cham. ISBN: 978-3-319-15490-9

# Free Vibration Analysis of Plate Assemblies using Dynamic Stiffness Method

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## Abstract

In this paper, the Dynamics Stiffness Method, which yields “exact” results for a certain class of plate structures, is applied to analyze free vibration of rectangular plates perpendicularly assembled. In order to obtain necessary results, the transformation matrices have been developed for three characteristic positions of rectangular plates. Numerical example is conducted for so-called L plate, consisting of two plates perpendicularly connected. In addition, numerical example is also conducted for so-called box, consisting of six rectangular plates perpendicularly connected. All plates have the same geometrical and mechanical properties. The accuracy of the results obtained by the Dynamics Stiffness Method is verified by comparing them with the solutions obtained by conventional Finite Element Method. The convergence of the computed natural frequencies is found to be rapid and excellent.

Keywords: Dynamics Stiffness Method, Plate Assemblies, Free vibration

## 1. Introduction

Plate assemblies are one of the basic structural components in civil engineering forming walls and floors of high-rise buildings, panels in ship hulls and aircraft fuselages. These structures are usually designed to low frequency excitations such as earthquakes and wind. The most common solution method for analysing these types of structures is the Finite Element Method (FEM). The FEM is the most versatile but also computationally expensive because many elements are needed to model accurately the mass and stiffness properties of large region. Also, the size of the finite element depends on the highest frequency in the analysis. The FEM may provide accurate dynamic characteristics of a structure if the wavelength is large compared to the mesh size. However, the finite element solution may become unreliable and inaccurate as the frequency increases. Although the accuracy can be improved by refining the mesh, this takes greater computer time and effort to solve two dimensional problems. As an alternative to the FEM in dynamic analysis the Dynamics Stiffness Method (DSM) can be used. The DSM is often referred to as an exact method as it is based on exact frequency-dependent shape functions obtained from the exact solution of the element differential equations of motion. Consequently, the dynamic stiffness matrix is frequency dependent, i.e. the analysis is performed in the frequency domain. The DSM is especially useful for one-dimensional elements where the accurate solutions for governing equations of motions are obtained. However, for two-dimensional elements, it is not possible to obtain exact solutions of partial differential equations that satisfy arbitrary boundary conditions. In order to find a solution of a problem, plate displacements are presented as infinite Fourier type series. For practical purposes, the series have to be truncated, which introduces an error. Consequently, the solutions are approximate and satisfy the prescribed degree of accuracy.

Earlier studies of free vibration characteristics of plates undergoing both transverse and in-plane vibration were conducted for Levy-type plates with two opposite edges simply supported [1-4]. The procedure for the development of the dynamic stiffness matrix for rectangular plate undergoing transverse and in-plane vibration with arbitrary boundary conditions can be found in the literature, [5-6], [7]. Bercin [8] employed the dynamic stiffness method to calculate free vibration characteristics of various directly coupled rectangular Levy-type plate assemblies. The dynamic stiffness matrix can be assembled in a completely analogous way to that used in the FEM. Thus, the assembly feature of the FEM can be applied to the dynamic stiffness matrix method.

The main objective of this paper is to present a general method for free vibration analysis of plate assemblies perpendicularly connected with arbitrarily assigned boundary conditions. The transformation matrix of rectangular plate is developed and the global dynamic stiffness matrix of plate assembly is obtained using the same assemblage procedure as in the FEM. The method is validated

against the available results in the literature, as well as against the results obtained using the FEM.

## 2. Dynamic stiffness matrix formulation

Basic procedure for obtaining the dynamic stiffness matrix of rectangular plate element will be presented in the following section.

The formulation of the dynamic stiffness matrix of a rectangular plate begins with the governing equations of motion. General form of the governing differential equation of motion of plate without presence of external load can be symbolically written as:

$$L_1(\mathbf{u}) = 0 \quad (1)$$

where  $L_1$  is a differential operator,  $\mathbf{u}$  is the corresponding displacement vector. The next step is to solve the above differential equation analytical for harmonically varying  $\mathbf{u}$  expressed as:

$$\mathbf{u} = \sum_{j=0}^{\infty} \hat{u}_j e^{i\omega_j t} \quad (2)$$

where  $\hat{u}_j$  represents amplitude of displacements for circular frequency  $\omega_j$ ,  $t$  is the time and  $i = \sqrt{-1}$ . From Eqs. (1) and (2) the equation of motion in the frequency domain is obtained:

$$L_2(\hat{\mathbf{u}}, \boldsymbol{\omega}) = 0 \quad (3)$$

where  $L_2$  is the differential operator defined in the frequency domain,  $\hat{\mathbf{u}}$  is the displacement vector in the frequency domain and  $\boldsymbol{\omega}$  is the corresponding circular frequency.

In order to find a solution of equation of motion, plate displacements are presented as infinite series:

$$\hat{u}(x, y) = \sum_{m=1}^{\infty} C_m f_m(x, y) \quad (4)$$

where  $C_m$  are integration constants and  $f_m(x, y)$  are base functions that satisfy Eq.(3).

Eq.(4) is presented in the form of infinite series, but for engineering purpose series have to be truncated to a point  $M$ . Now, Eq. (4) becomes:

$$\hat{u}(x, y) \approx \sum_{m=1}^M C_m f_m(x, y) \quad (5)$$

In order to obtain the dynamic stiffness matrix, the relation between the displacement and force vectors has to be found. Displacements and forces along plate boundaries are spatially dependant values. In order to avoid the spatial dependence, the Projection method [5-6] is used. The displacements and forces are projected onto a set of projection functions  $h(s)$ :

$$\begin{aligned} \hat{q}(s) &\approx \sum_{n=1}^M \langle \hat{q}, h_n \rangle h_n(s) \\ \hat{Q}(s) &\approx \sum_{n=1}^M \langle \hat{Q}, h_n \rangle h_n(s) \end{aligned} \quad (6)$$

where  $\tilde{q}_n = \langle \hat{q}, h_n \rangle = \frac{2}{L} \int_s \hat{q}(s) \cdot h_n(s) ds$  is the projection of displacements and  $\tilde{Q}_n = \langle \hat{Q}, h_n \rangle = \frac{2}{L} \int_s \hat{Q}(s) \cdot h_n(s) ds$  is the projection of forces along the plate boundary  $L$ . These projections are collected into the corresponding vectors:

$$\tilde{\mathbf{q}} = [\langle \hat{q}, h_n \rangle] \quad \tilde{\mathbf{Q}} = [\langle \hat{Q}, h_n \rangle] \quad (7)$$

Now, it is possible to define the relation between the projections of displacements and forces for transverse  $\tilde{\mathbf{K}}_{Dt}$  and in plane  $\tilde{\mathbf{K}}_{Di}$  vibrations. In the case of linear theory these two conditions are independent and the dynamic stiffness matrix for rectangular plate using the DSM can be presented as:

$$\tilde{\mathbf{K}}_D = \begin{bmatrix} \tilde{\mathbf{K}}_{Dt} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{K}}_{Di} \end{bmatrix} \quad (8)$$

The dynamic stiffness matrix gives the relation between the projections of the displacement vector  $\tilde{\mathbf{q}}$  and the force vector  $\tilde{\mathbf{Q}}$ :

$$\tilde{\mathbf{Q}} = \tilde{\mathbf{K}}_D \tilde{\mathbf{q}} \quad (9)$$

The projection of the force vector  $\tilde{\mathbf{Q}}$  is defined as:

$$\begin{aligned}
\tilde{\mathbf{Q}}^T &= [\tilde{\mathbf{Q}}_t \quad \tilde{\mathbf{Q}}_i] \\
\tilde{\mathbf{Q}}_t^T &= [\tilde{\mathbf{Q}}_{0,t} \quad \tilde{\mathbf{Q}}_{1,t} \quad \cdots \quad \tilde{\mathbf{Q}}_{m,t} \quad \cdots \quad \tilde{\mathbf{Q}}_{M,t}] \\
\tilde{\mathbf{Q}}_{m,t}^T &= [{}^1\tilde{\mathbf{Q}}_{m,t}^T \quad {}^2\tilde{\mathbf{Q}}_{m,t}^T \quad {}^3\tilde{\mathbf{Q}}_{m,t}^T \quad {}^4\tilde{\mathbf{Q}}_{m,t}^T] \\
{}^j\tilde{\mathbf{Q}}_{0,t}^T &= [{}^j\bar{T}_{x_{S_0}} \quad {}^jM_{x_{S_0}}] \quad {}^k\tilde{\mathbf{Q}}_{0,t}^T = [{}^j\bar{T}_{y_{S_0}} \quad {}^jM_{y_{S_0}}] \\
{}^j\tilde{\mathbf{Q}}_{m,t}^T &= [{}^j\bar{T}_{x_{S_m}} \quad {}^j\bar{T}_{x_{A_m}} \quad {}^jM_{x_{S_m}} \quad {}^jM_{x_{A_m}}] \\
{}^k\tilde{\mathbf{Q}}_{m,t}^T &= [{}^k\bar{T}_{y_{S_m}} \quad {}^k\bar{T}_{y_{A_m}} \quad {}^kM_{y_{S_m}} \quad {}^kM_{y_{A_m}}]
\end{aligned} \tag{10}$$

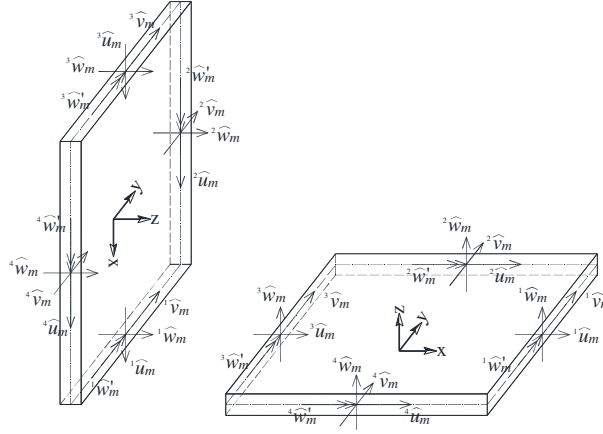
$$\begin{aligned}
\tilde{\mathbf{Q}}_i^T &= [\tilde{\mathbf{Q}}_{0,i} \quad \tilde{\mathbf{Q}}_{1,i} \quad \cdots \quad \tilde{\mathbf{Q}}_{m,i} \quad \cdots \quad \tilde{\mathbf{Q}}_{M,i}] \\
\tilde{\mathbf{Q}}_{m,i}^T &= [{}^1\tilde{\mathbf{Q}}_{m,i}^T \quad {}^2\tilde{\mathbf{Q}}_{m,i}^T \quad {}^3\tilde{\mathbf{Q}}_{m,i}^T \quad {}^4\tilde{\mathbf{Q}}_{m,i}^T] \\
\tilde{\mathbf{Q}}_{0,i}^T &= [{}^1N_{x_{S_0}} \quad {}^2N_{y_{S_0}} \quad {}^3N_{x_{S_0}} \quad {}^4N_{y_{S_0}}] \\
{}^j\tilde{\mathbf{Q}}_{m,i}^T &= [{}^jN_{x_{S_m}} \quad {}^jN_{x_{A_m}} \quad {}^jN_{xy_{S_m}} \quad {}^jN_{xy_{A_m}}] \\
{}^k\tilde{\mathbf{Q}}_{m,i}^T &= [{}^kN_{y_{S_m}} \quad {}^kN_{y_{A_m}} \quad {}^kN_{xy_{S_m}} \quad {}^kN_{xy_{A_m}}]
\end{aligned}$$

where  $j = 1, 3$  and  $k = 2, 4$  represent the corresponding edges of rectangular plate element, Fig.1. Similarly, the projection of the displacement vector  $\tilde{\mathbf{q}}$  is defined as:

$$\begin{aligned}
\tilde{\mathbf{q}}^T &= [\tilde{\mathbf{q}}_t \quad \tilde{\mathbf{q}}_i] \\
\tilde{\mathbf{q}}_t^T &= [\tilde{\mathbf{q}}_{0,t} \quad \tilde{\mathbf{q}}_{1,t} \quad \cdots \quad \tilde{\mathbf{q}}_{m,t} \quad \cdots \quad \tilde{\mathbf{q}}_{M,t}] \\
\tilde{\mathbf{q}}_{m,t}^T &= [{}^1\tilde{\mathbf{q}}_{m,t} \quad {}^2\tilde{\mathbf{q}}_{m,t} \quad {}^3\tilde{\mathbf{q}}_{m,t} \quad {}^4\tilde{\mathbf{q}}_{m,t}] \\
{}^j\tilde{\mathbf{q}}_{0,t}^T &= [{}^jW_{S_0} \quad {}^jW'_{S_0}] \\
{}^j\tilde{\mathbf{q}}_{m,t}^T &= [{}^jW_{S_m} \quad {}^jW_{A_m} \quad {}^jW'_{S_m} \quad {}^jW'_{A_m}]
\end{aligned} \tag{11}$$

$$\begin{aligned}
\tilde{\mathbf{q}}_i^T &= [\tilde{\mathbf{q}}_{0,i} \quad \tilde{\mathbf{q}}_{1,i} \quad \cdots \quad \tilde{\mathbf{q}}_{m,i} \quad \cdots \quad \tilde{\mathbf{q}}_{M,i}] \\
\tilde{\mathbf{q}}_{0,i}^T &= [{}^1u_{S_0} \quad {}^2v_{S_0} \quad {}^3u_{S_0} \quad {}^4v_{S_0}] \\
\tilde{\mathbf{q}}_{m,i}^T &= [{}^1\tilde{\mathbf{q}}_{m,i} \quad {}^2\tilde{\mathbf{q}}_{m,i} \quad {}^3\tilde{\mathbf{q}}_{m,i} \quad {}^4\tilde{\mathbf{q}}_{m,i}] \\
{}^j\tilde{\mathbf{q}}_{m,i}^T &= [{}^ju_{S_m} \quad {}^ju_{A_m} \quad {}^jv_{S_m} \quad {}^jv_{A_m}]
\end{aligned}$$

where  $j = 1, 2, 3, 4$ . Represented vectors are given in the local coordinate system, Fig.1.

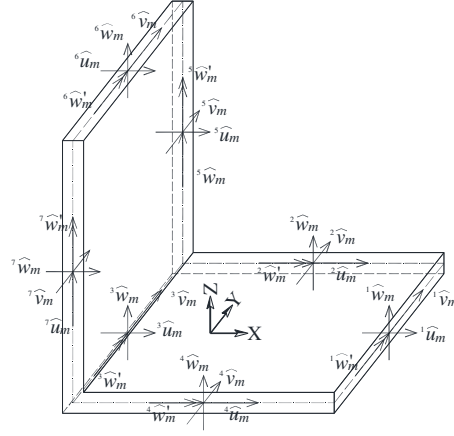


**Fig. 1:** Edge displacements of L plate in local coordinate system

For accurate results in the dynamic stiffness method one element is sufficient, regardless the dimensions, the highest frequency and the boundary conditions of rectangular plate.

### 3. Transformation of dynamic stiffness matrix

As shown in the previous section, in the DSM the force-displacement relation of rectangular plate is defined by the dynamic stiffness matrix. Therefore, the same assemblage procedure is used as in the FEM. In order to assemble rectangular plates, transformation of displacements and forces along the edges from local coordinate system to global coordinate system is necessary. Fig.1 shows the system consisting of two plates, so-called L plate, with edge displacements in the local coordinate systems. Transformed displacements and forces along edges in the global coordinate system are presented in Fig.2.



**Fig. 2.** Edge displacements of L plate in global coordinate system

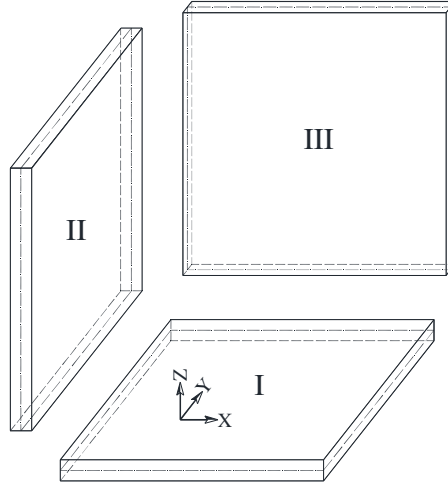
Using the transformation matrix the projections of the displacement and force vectors have been transformed from local to global coordinate system:

$$\begin{aligned}
 \tilde{\mathbf{q}}_T &= \mathbf{T}_T^T \tilde{\mathbf{q}} \\
 \tilde{\mathbf{Q}}_T &= \mathbf{T}_T^T \tilde{\mathbf{Q}} \\
 \tilde{\mathbf{Q}}_T &= \mathbf{T}_T^T \tilde{\mathbf{K}}_D \mathbf{T}_T \tilde{\mathbf{q}}_T = \tilde{\mathbf{K}}_{D_T} \tilde{\mathbf{q}}_T
 \end{aligned} \tag{12}$$

where  $\mathbf{T}_T$  is the transformation matrix, vectors  $\tilde{\mathbf{q}}_T$  and  $\tilde{\mathbf{Q}}_T$  are the projections of the displacement and force vectors in the global coordinate system and  $\tilde{\mathbf{K}}_{D_T}$  is the dynamic stiffness matrix of plate element in the global coordinate system.

For perpendicularly assembled rectangular plates three positions can be available, Fig.3. Each plate position requires transformation to the global coordinate system.





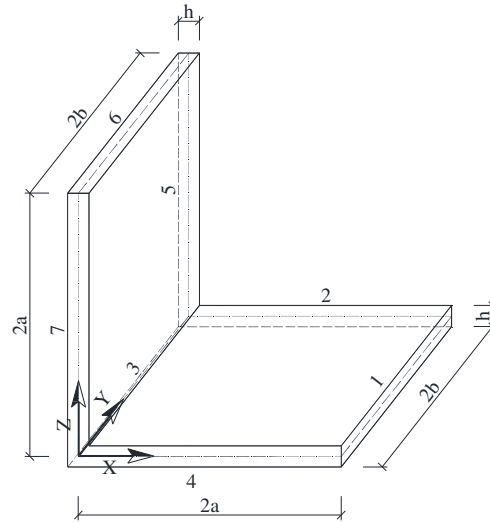
**Fig. 3.** Plate positions

#### 4. Numerical examples

As illustrative examples two plate assemblies will be considered, L plate consisting of two perpendicularly assembled rectangular plates and box consisting of six perpendicularly assembled rectangular plates. All plates have the same geometrical and material properties, Young's modulus  $E = 30\text{GPa}$ , mass density  $\rho = 2.5\text{ t/m}^3$ , Poisson's ratio  $\nu = 0.15$ , edge length  $a = b = 3\text{ m}$  and plate thickness  $h = 0.15\text{ m}$ . As mentioned earlier boundary conditions can be chosen arbitrarily. In order to illustrate the convergence and accuracy of the dynamic characteristics of the systems, the first 10 natural frequencies computed using the proposed method are compared with the results obtained using SAP2000, [9].

##### L plate

L plate assembly consisting of two rectangular plates perpendicularly assembled is presented in Fig.4. Edges  $1$  and  $6$  are simply supported (plate displacements in three orthogonal directions are zero) while all other edges are free. Natural frequencies have been computed for different number of terms in  $M$  in the general solution and are given in Table 1.



**Fig. 4.** L plate with dimensions and edge numeration

It can be seen that high convergence and accuracy has been achieved using only 3-5 terms in the general solution using the proposed method. Natural frequencies computed using SAP2000 for different mesh sizes have been presented in Table 2. As the number of finite elements increases the results converge to the present solution based on the DSM.

**Table 1.** Natural frequencies (in Hz) of L plate computed using the proposed method

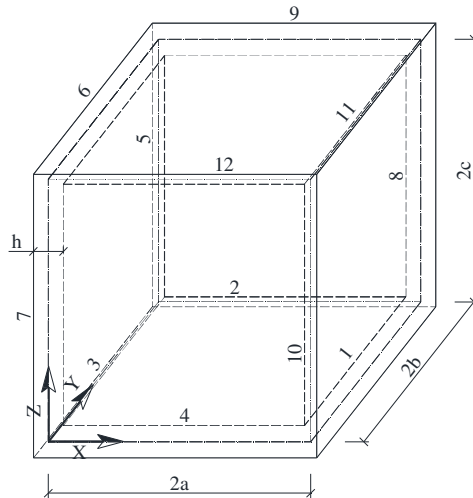
Mode No.	$M = 1$	$M = 2$	$M = 3$	$M = 4$	$M = 10$
1	26.3	26.3	26.3	26.3	26.3
2	40.9	40.9	40.9	40.9	41.0
3	43.9	45.2	45.5	45.6	45.7
4	55.7	56.6	56.8	56.9	56.9
5	98.4	100.5	101.0	101.3	101.5
6	104.6	104.6	104.6	104.6	104.6
7	106.7	108.5	109.0	109.1	109.3
8	123.2	121.6	121.3	121.1	121.0
9	126.7	127.6	127.9	128.0	128.1
10	130.3	130.2	130.2	130.2	130.2

**Table 2.** Natural frequencies (in Hz) of L plate computed using SAP2000

Mode No.	Number of finite elements				
	5x5	10x10	20x20	40x40	100x100
1	26.3	26.3	26.3	26.3	26.3
2	40.9	41.0	41.0	41.0	41.0
3	43.8	45.3	45.6	45.7	45.7
4	54.3	56.3	56.8	56.9	56.9
5	89.0	98.2	100.8	101.4	101.6
6	94.9	104.6	104.6	104.6	104.6
7	104.6	105.4	108.4	109.1	109.3
8	119.8	120.8	121.0	121.0	121.0
9	121.8	126.6	127.7	128.0	128.1
10	129.5	130.2	130.2	130.2	130.2

**Box**

Box presented in Fig.5 consists of six rectangular plates perpendicularly assembled. Edges 1 and 3 are clamped while all others are free.

**Fig. 5.** Box with dimensions and edge numeration

The first ten natural frequencies computed using the proposed method and the FEM are presented in Tables 3 and 4, respectively. Again, excellent agreement has

been achieved between the present solution and the solution obtained from the FEM.

**Table 3.** Natural frequencies (in *Hz*) of Box computed using the proposed method

Mode No.	$M = 1$	$M = 2$	$M = 3$	$M = 4$	$M = 5$	$M = 10$
1	59.6	60.9	61.6	62.1	62.4	62.9
2	64.1	65.7	65.8	65.8	65.8	65.9
3	65.4	66.2	67.0	67.3	67.5	67.8
4	68.2	68.6	68.7	68.7	68.7	68.7
5	84.6	84.7	84.8	84.8	84.8	84.8
6	85.6	87.0	87.6	88.0	88.3	88.7
7	92.3	94.3	95.3	95.4	95.4	95.4
8	95.4	95.4	95.4	95.8	96.1	96.6
9	118.0	121.7	112.5	122.8	122.9	123.1
10	138.0	142.1	142.2	142.3	142.3	142.3

**Table 4.** Natural frequencies (in *Hz*) of Box computed using SAP2000

Mode No.	Number of finite elements				
	5x5	10x10	20x20	40x40	100x100
1	62.6	63.0	63.0	63.0	63.0
2	63.4	65.2	65.7	65.8	65.9
3	66.0	67.7	67.9	67.9	67.9
4	67.0	68.0	68.5	68.6	68.7
5	81.1	83.8	84.5	84.7	84.8
6	87.0	88.2	88.6	88.8	88.8
7	90.9	94.2	95.1	95.4	95.4
8	94.5	96.2	96.6	96.7	96.7
9	120.8	122.7	123.1	123.2	123.2
10	133.8	140.2	142.8	142.2	142.3

## 5. Conclusion

Application of the DSM in the free vibration analysis of perpendicularly connected plate assemblies having arbitrary boundary conditions has been presented in this paper. The dynamic stiffness matrix of plate element has been transformed

from local to global coordinate system using the corresponding transformation matrix and then assembled using the same assemblage procedure as in the FEM. The free vibration analysis of L plate and Box has been carried out using the proposed method. The DSM provides very accurate solutions when compared with the conventional FEM. In addition, the convergence of the proposed method is rapid, as only 3-5 terms in the general solution are sufficient to obtain accurate results. The proposed method has demonstrated high accuracy and low computational time in comparison with the FEM, especially in the high frequency range.

#### Acknowledgments

The authors are grateful to the Ministry of Science and Technology, Republic of Serbia, for the financial support of this research within the Project TR 36046.

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