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COMPOSITE POLYHEDRAL FORMS OBTAINED BY COMBINING CONCAVE PYRAMIDS OF THE SECOND SORT WITH ARCHIMEDEAN SOLIDS

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ABSTRACT

The paper discusses a possibility of forming composite polyhedral forms using, as an outline, geometry of certain Archimedean solids, compatible with the geometry of concave pyramids of the second sort (CP II). Given that CP II as its base may have polygon from n=6 to n=9, with the possibility of forming the lateral sheet even with the decagonal base, Archimedean solids which could be taken into consideration are those which contain hexagon, octagon or decagon among their polygonal faces. These are: the truncated tetrahedron, truncated cube, truncated octahedron, truncated cuboctahedron, truncated dodecahedron, truncated icosahedron and truncated icosidodecahedron. These seven solids are augmented by adding: CP II-n-M, CP II-n- m, or CP II – n-B. In the typical case, if we adhere to the criterion that only one type of pyramid is added to one side of the body, such an augmentation would provide 21 new polyhedra. The factual number of the variations of the concave polyhedra obtained by adding different types of CP II onto the mentioned Archimedean solids is far greater, so the paper also deals with the possible number of these polyhedral shapes. In this manner, it is possible to get deltahedral concave forms, and the abundance of forms resulting from combinations of these polyhedra can serve for further research in geometry, design and structural fields.

Keywords: concave polyhedra, composite polyhedra, Archimedean solids, CP II, augmentation

SUBJECT CODE: Theoretical Geometry

INTRODUCTION

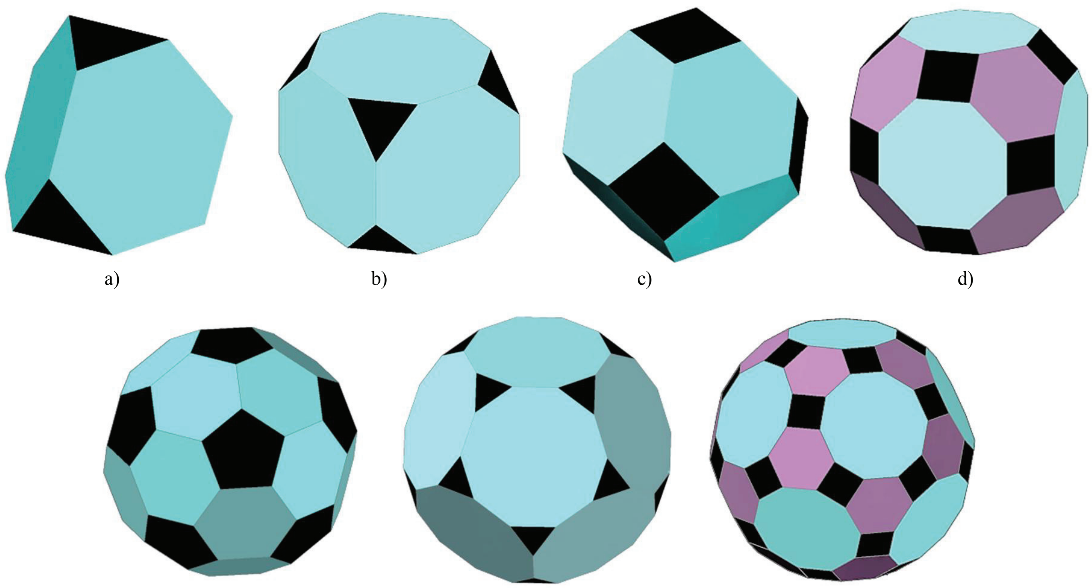
The paper examines a possibility of forming composite polyhedral forms comprised of concave pyramids of the second sort (CP II), using geometry of Archimedean solids as an outline. In this regard, it is a continuation of research in [6], [7], [9], [10] conducted on the concave cupolae of the second and higher sorts. The geometrical characteristics and the occurrence of CP II - Type A are discussed in detail in [11], [8], and the characteristics of CP II - Type B are discussed in [12], [8]. Overall, CP II arise by folding and corrugating a planar triangular net, in order to obtain spatial pentahedral cells arranged in a polar array around the normal axis through the centroid of the regular n-sided basis, forming a deltahedral lateral sheet. In this manner, the mentioned two types of CP II may occur: CP II- type A, with 5n equilateral triangles in the lateral sheet, which then may be assembled in such a way to create the lateral sheet with greater height (CP II- n-M) or with lesser height (CP II–n-m), and CP II - type B with 3n equilateral triangles in the lateral sheet, which produces only one shape and height of such a solid for the given polygonal base.

In this research we will deal with the formation of new polyhedral structures from an engineering point of view as done in [1], [5], [10], only brushing upon group theory implications in order to link this study with possible future directions of research. The focus of the research is on the possibilities of obtaining new polyhedral shapes whose versatility and geometric distinctness can serve as a starting point fortheir possible applications in fields of science, engineering or design.

1. DETERMINATION OF THE POSSIBLE NUMBER OF DIFFERENT SHAPES OBTAINED BY AUGMENTATION OF ARCHIMEDEAN SOLIDS BY CP II

Given that CP II can have, as its base, a polygon from n=6 to n=9, with the possibility of forming the lateral sheet for as many as n=10, Archimedean solids that could be taken into consideration for the augmentations with CP II are those whose polygonal faces include hexagons, octagons and decagons. These solids are: truncated tetrahedron, truncated cube, truncated octahedron, truncated cuboctahedron, truncated icosahedron, truncated dodecahedron and truncated icosidodecahedron (Fig. 1).

These seven solids can be augmented by joining: CP II-n-M/m, or CP II–n-B onto their congruent polygonal faces. In the most typical case, if we adhere to the criterion that only one type of concave pyramid can be bijectively added to a single solid (i.e. *K* concave pyramids on *K* compatible polygonal faces), by such augmentations we would get 7·3 = 21 new polyhedra. They are shown in the gallery in the oncoming section of the paper. The actual number of permutations which cover all possible types and positions of CP II is far greater, so we give a brief overview of the possible number of different polyhedral shapes obtained in this manner.



a) b) c) d)

e) f) g)

Fig. 1: Seven Archimedean solids that may be augmented by CP II:
a) truncated tetrahedron, b) truncated cube, c) truncated octahedron, d) truncated cuboctahedron,
e) truncated icosahedron f) truncated dodecahedron g) truncated icosidodecahedron

Using this method, it is possible to obtain versatile concave deltahedral forms, and the abundance of shapes originated by combining these polyhedra may serve as a source for the further investigations beyound the scope of geometry and design.

It is possible to form a polyhedron that uses Archimedean solids as a foundation on whose polygonal faces bases of CP II can be joined, taken that the corresponding faces of these solids are congruent. We can place CP II in the external area, making augmentation of Archimedean solids, or in the interior space, making incavations. In this paper, due to limited space, we will present only the most typical examples of Archimedean solids' augmentation using CP II.

Given that the minimal n-sided basis over which CP II may be formed is hexagon, and the maximal is decagon, we will adjoin the bases of CP II only to hexagonal, octagonal or decagonal faces of the Archimedean solids. The ones which contain hexagons in their composition are: truncated tetrahedron (Fig. 1 a), truncated octahedron (Fig. 1 c), truncated cuboctahedron (Fig. 1 d), truncated icosahedron (Fig. 1 e) and the truncated icosidodecahedron (Fig. 1 g), so they can be augmented using CP II-6-M, CP II-6-m or CP II-6-B. Archimedean solids which contain octagons are: truncated cube (Fig. 1 b) and truncated cuboctahedron (Fig. 1 d), so they can be augmented using CP II-8-M, CP II-8-m and CP II-8-B. Decagons are present in: truncated dodecahedron (Fig. 1 f) and truncated icosidodecahedron (Fig. 1 g), so they can be augmented using segments - lateral sheets: $\overline{\mathrm{CP}}$ II-10-M, $\overline{\mathrm{CP}}$ II-10-m and $\overline{\mathrm{CP}}$ II-10-B. As shown in Fig. 1, there are two solids – truncated cuboctahedron and truncated icosidodecahedron which could be augmented using two different types of CP II: truncated cuboctahedron with CP II-6 and CP II-8, while truncated icosidodecahedron can be augmented with CP II-6 and $\overline{\text{CP}}$ II-10.

Let us examine the possible number of different composite polyhedra obtained by augmentation of the Archimedean solids by CP II. The actual number is obtained by taking into account all the possible permutations of the three types of CP II, plus the fourth case of blank face, which is not augmented. Thus, when cases of augmentation are concerned (without incavations), this problem can be reduced to the problem of determining the orbit of the group action on a set of sides of the chosen Archimedean solid. To simplify, the problem comes down to the problem of determining in how many ways we can paint observed polygonal faces of the given solid by using min 4 colors, i.e. in the case of the solids with two types of the basis onto which CP II can be added, using 3 + 3 + 1 = 7 colors.

This is the case of simple application of Burnside's Lemma [1], given in the following formula:

$$|X/G| = \frac{1}{|G|}\sum_{g\in G}|X^g| \qquad \textit{(Eq. 1)}$$

where G is a finite group that acts on a set X, and g is an element of the group G,

Or more specifically, we use extention of Burnside's Lemma, The Pólya enumeration theorem [3], also known as the Redfield–Pólya Theorem, which allows counting of discrete combinatorial objects as a function of their 'order' [13].

In this paper, we will give just an example of a solution to this problem, pertaining to the truncated cube. It has six octagonal faces and belongs to octahedral point (symmetry) group (Oh), i.e. order of the rotation group is the same as for the cube, and stands at 24. So, due to the identity of the problems, it comes down to the possible ways to paint faces of the cube using ***m*** colors. For ***m*** possible colors that could be used in coloring the cube, the general formula (for cube) is:

$$\frac{1}{24}(m^6 + 3m^4 + 12m^3 + 8\,m^2) = 240\ [4] \qquad \textit{(Eq. 2)}$$

If we now replace the value and introduce $\boldsymbol{m}=4$, the number of possible combinations is:

$$\frac{1}{24}(4^6 + 3\cdot 4^4 + 12\cdot 4^3 + 8\ \cdot 4^2) = 240 \qquad \text{(Eq. 3)}$$

From the given examples, we see that the number of the new solids obtained just by augmenting one or all faces of truncated cube, amounts to 240. If we add the possible incavations to this number, i.e. denting CP II to the interior space of the truncated cube, now we would take into account 7 different 'cases', equivalent to the problem of colors by which we might paint faces of a cube, so the number would grow to 5390, according to the general formula given above.

For the remaining cases of augmented Archimedean solids onto whose faces we can add 3 types of CP II, the result can be obtained in a similar manner - by application of Pólya enumeration theorem, whose principle is shown above. Due to the lack of the space in the paper, we will not engage in more detailed explanation of each case.

Thus, the aggregate number of such permutations would far exceed the possibilities of practical systematization, so we will comply with the criterion of commonality of CP II types that augment one solid. In this case, we would obtain 3 new concave composite polyhedra for each of the listed Archimedean solids, with the criterion that every available face congruent to the CP II base is augmented with the identical type of the adequate CP II. Assumably, if we introduce the possibility that on the Archimedean solids which can be augmented by two different n-side based CP II (truncated cuboctahedron, truncated icosidodecahedron) we may combine types of both CP II used (M, m and B), the number of the new solids, as the number of variations with repetition of class k = 2, with n = 3 elements, would then increase from 3 to:

$$\overline{V_3^2} = 3^2 = 9 \qquad \text{(Eq. 4)}$$

Hence, we would get: 3·5+2·9= 33 new concave composite polyhedra obtained in this manner.

2. TYPES OF CP II AND THEIR POTENTIAL AS BUILDING BLOCKS OF A COMPOSITE POLYHEDRON

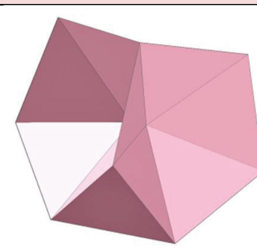
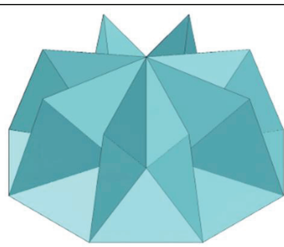
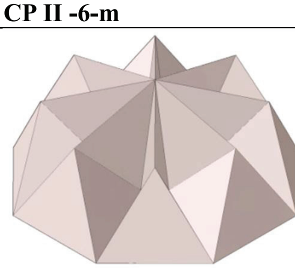
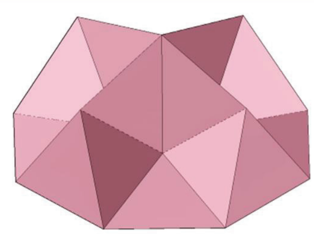
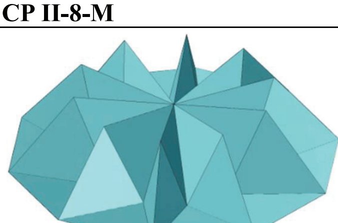
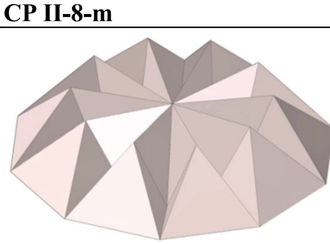
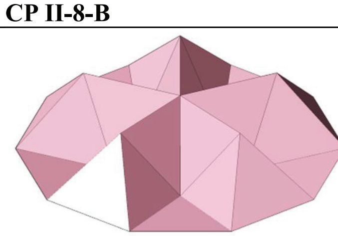
Considering that the properties of CP II type A are described in [11], the properties of CP II type B in [12], and both briefly recapitulated in the introduction, in this section we examine the main differences between these types and ways of their integration into the geometry of Archimedean solids. Although the key difference between these two types of CP II is that in the type CP II–n-A the base polygon (starting point for the formation of the lateral sheet) can be any polygon between n=6 and n=10, while in CP II-n-B it can be only even sided polygon, in this case this fact does not play an important role, because the only those CP II formed over the even-numbered sided polygons can be used (as shown in Table 1- blue and beige). CP II-n-A, due to the manner of forming and the number of triangles in the mantle, has 2n planes of symmetry, in contrast to the CP II-n-B (shown Table 1 - pink) which has n planes of symmetry. Therefore,) there is only one possible way of joining CP II-n-A (CP II-n-M and CP II-n-m onto the faces of Arcimedean solids, while CP II-n-B provides more opportunities, because every single CP II can rotate around the normal axis through the centroid of the basis for the angle of 0° or π /n, where once again we have an identical problem of 'painting' faces, this time using two colors - one for each of these two positions. So, in the case of the truncated cube we get the actual number of:

$$\frac{1}{24}(2^6 + 3\cdot 2^4 + 12\cdot 2^3 + 8\cdot 2^2) = 10 \qquad \text{(Eq. 5)}$$

possible various solids, just by augmentation of truncated cube using CP II-8-B. Thus, the total number of possible augmentations of truncated cube only by the same type of CP II-8, increases to 1 + 1 + 10 = 12.

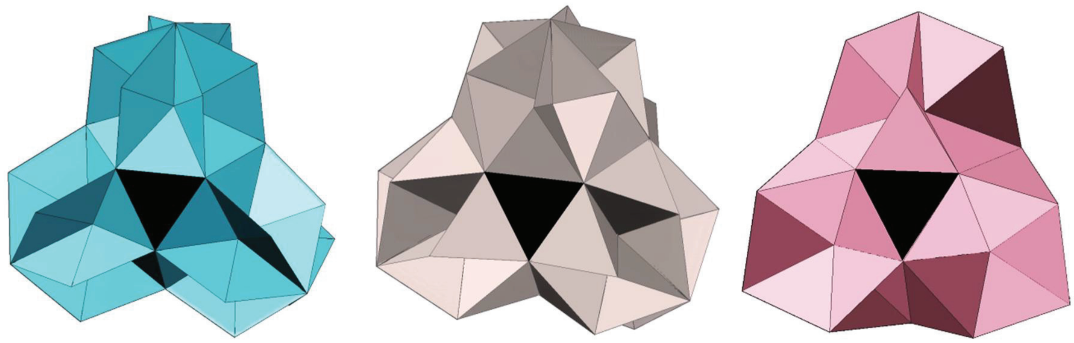
In this paper we present only those cases which imply that all the faces of a solid are augmented by the same type of CP II, and if it is CP II-n-B, that all of them are identicly oriented, for a clearer overview of typical shapes.

Table 1: types of CP II

type sides	**Type A CP II-n-M**	**Type A CP II-n-m**	**Type B CP II-n-B**
n=6			
	CP II-6-M	**CP II -6-m**	**CP II-6-B**
n=8			
	CP II-8-M	**CP II-8-m**	**CP II-8-B**
n=10			
	CP II-10-M	**CP II-10-m**	**CP II-10-B**

3. GALLERY OF COMPOSITE POLYHEDRA FORMED BY AUGMENTING ARCHIMEDEAN SOLIDS BY CP II

Hereafter, we show a gallery of typical representatives of concave composite polyhedra formed by augmentation of Archimedean solids using just one type of CP II, number of which suits the bijective mapping to the observed solid's congruent faces. To facilitate monitoring, the Archimedean solids augmented using CP II-n-M are shown in blue, using CP II-n-m in beige, while those using CP II-n-B are shown in pink, as in the Table 1. The faces of the Archimedean solids that were not eligible for augmentation by CP II (triangular, rectangular and pentagonal) are retained and displayed in black.

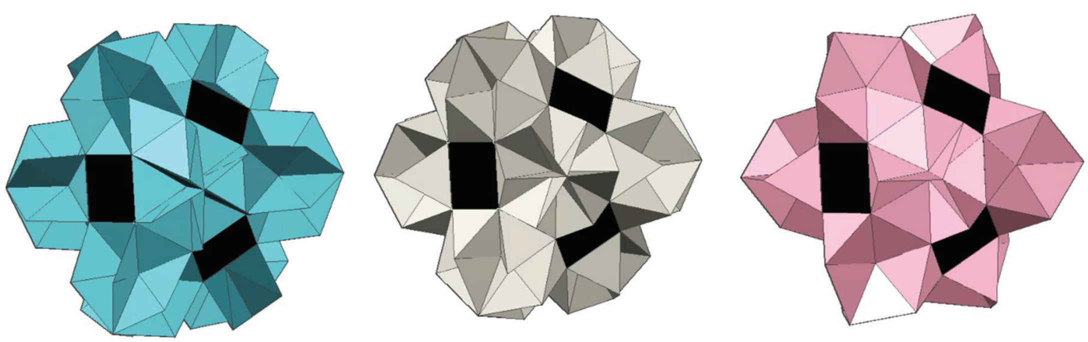


a) b) c)

Fig. 2: Composite polyhedra obtained by augmentation of Truncated tetrahedron and CP II-6:
a) by CP II-6-M, b) by CP II--6m, c) by CP II-6-B

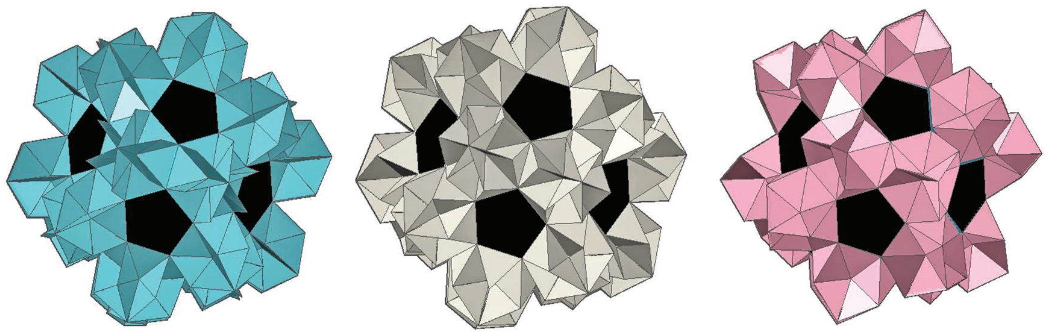
a) b) c)

Fig. 3: Composite polyhedra obtained by augmentation of Truncated Cube and CP II-8:
a) by CP II-8-M, b) by CP II--8m, c) by CP II-8-B



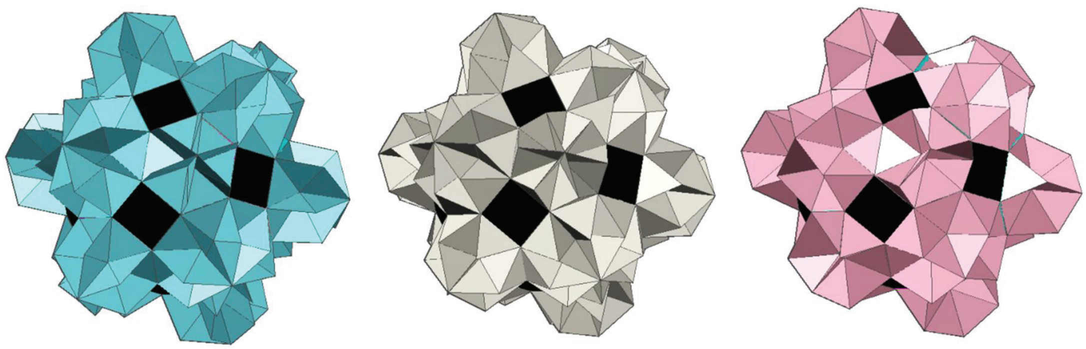
a) b) c)

Fig. 4: Composite polyhedra obtained by augmentation of Truncated octahedron and CP II-6:
a) by CP II-6-M, b) by CP II-6-m, c) by CP II-6-B



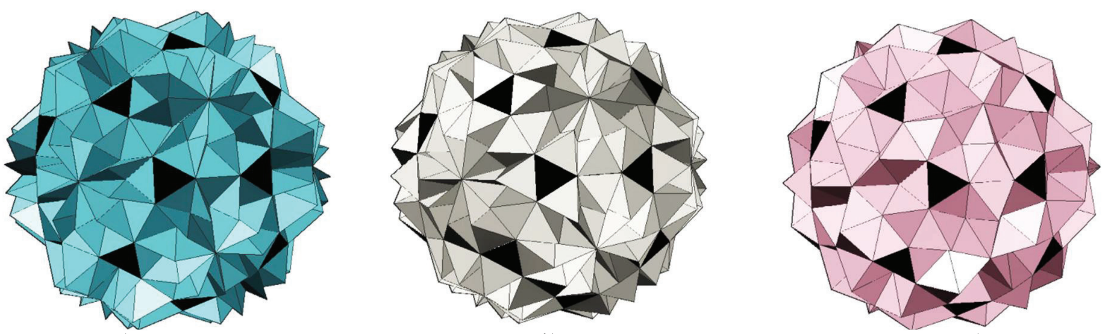
a) b) c)

Fig. 5: Composite polyhedra obtained by augmentation of Truncated icosahedron and CP II-6:
a) by CP II-6-M, b) by CP II-6-m, c) by CP II-6-B



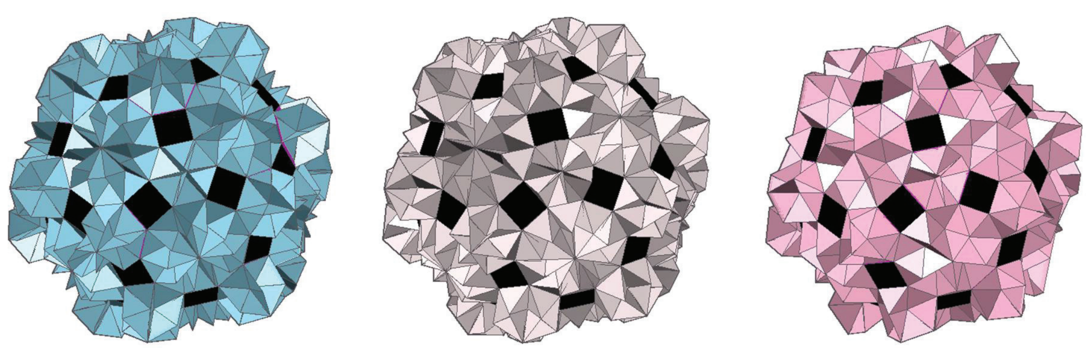
a) b) c)

Fig. 6: Composite polyhedra obtained by augmentation of Truncated Cuboctahedron by CP II-6 and CP II-8:
a) by CP II-6-M and by CP II-8-M, b) by CP II-6-m and by CP II-8-m, c) by CP II-6-B and by CP II-8-B



a) b) c)

Fig. 7: Composite polyhedra obtained by augmentation of truncated dodecahedron and CP II-10
a) by CP II-10-M, b) by CP II -10-m, c) by CP II-10-B



a) b) c)

Fig. 8: Composite polyhedra obtained by augmentation of icosidodecahedron by CP II 6 and CP II-10:
a) by CP II-6-M and by CP II-10-M, b) by CP II-6-m and by CP II-10-m, c) by CP II-6-B and by CP II-10-B

We will point out that among the examples given, there are deltahedra, i.e. polyhedra whose all the faces are identical equilateral triangles. These solids are created by augmentations of truncated tetrahedron (Fig. 2), truncated cube (Fig. 2) and truncated dodecahedron (Fig. 7). Given that they consist of identical elements - equilateral triangles, we might continue to explore whether and which of these solids could be assembled from a planar, unbroken net.

The obtained concave composite solids consisting of regular polygons other than equilateral triangles, such as squares and pentagons can also be converted into deltahedra by additional augmentations, using four-sided and five-sided pyramids, respectively.

CONCLUSIONS

The paper describes the procedure for obtaining composite concave polyhedra by augmentations of Archimedean solids using concave pyramids of the second sort (CP II), whereby we also discussed the possible number of the newly obtained solids. The gallery of the most typical examples of such augmentation is given. The aim of the paper is, in addition to the presentation of the formation method, to provide a visual insight as the initial part of the overall analysis, in order to make comparisons and receive information on the possible shapes of the newly formed solid. Consequently, we might include some other criteria in the process of each solid's evaluation in terms of their possible application, for example: dihedral angles' values, area and volume ratio, photo-illumination of the solids' faces and their relation with the total area of the body, the effective drainage methods, and the like, all of which can be starting points for the future research. Based on the forms displayed in this paper, we may also take into consideration certain fragments of these solids and examine their potential for further application.

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