6th International Congress of Serbian Society of Mechanics
Mountain Tara, Serbia, June 19-21, 2017

STABILITY ANALYSIS OF VERTICAL CUTS

Andrija Radović[1], Nikola Blagojević[2], Dragoslav Šumarac[3]

[1] Faculty of Civil Engineering
The University of Belgrade, Bul. kralja Aleksandra 73, 11000 Belgrade
E-mail: andrijaradovic1@gmail.com
[2] Faculty of Civil Engineering,
The University of Belgrade, Bul. kralja Aleksandra 73, 11000 Belgrade
E-mail: nblagojevic@grf.bg.ac.rs
[3] Faculty of Civil Engineering,
The University of Belgrade, Bul. kralja Aleksandra 73, 11000 Belgrade
E-mail: sumi@eunet.rs

Abstract:

Stability of vertical cuts is analyzed using kinematic theorem. It is assumed that there is no internal friction in the soil (φ=0) and Coulomb's yield criterion is used for determining the failure of the soil mass. Limit analysis theorem is briefly explained and a short review of existing upper and lower bound solutions is given. In this paper, new upper bound solutions are obtained by calculating the stability number N_s for assumed failure patterns. First, a failure pattern consisting of a circular arc and a straight line is analyzed. Then, a method for calculating the stability number of a failure pattern defined by a function is presented. Failure patterns defined with a sinus function, second degree polynomial and exponential function are considered in this paper. For each of the assumed failure patterns, stability number is calculated using the method presented here. The lowest value of the stability number ($N_s = 3.84$) is obtained when failure pattern is defined using exponential function.

Key words: stability, vertical cuts, upper bound solution, kinematic theorem

1. Introduction

Determining the stability of a vertical cut is a problem of great importance in civil engineering, since it is commonly present in every day practice. Loss of stability of an unsupported vertical cut can have large consequences, from endangering objects functionality and stability to loss of human life. In order to define the point at which vertical cut reaches instability due to the weight of the soil, it is necessary to know the critical height of the vertical cut. Many attempts were made in order to find the solution for critical height of a vertical cut, but still no accurate solutions has been found [1]. The solution to this problem can be divided into two parts, solving the problem using upper bound (kinematic theorem) solution and lower bound (static theorem) solution [2]. The accurate solution lies between two values obtained by this two theorems. In this paper a short overview of solutions proposed so far will be given. Also, new upper bound solutions will be presented and compared with already existing upper bound solutions. The theory behind this approach will also be explained.

2. Statement of the problem

In order to apply limit analysis theorem, mechanical behavior of soil must be assumed by defining stress-strain relationships for the materials as well as yield criterion. In this paper, soil is assumed to be perfectly plastic, while Coulomb's yield criterion is used for determining failure of the soil. It is assumed that there is no internal friction in the soil (ϕ=0), so yield criterion is the same as Tresca's failure criterion. Cohesion of the soil is ***c***.

Equation that defines critical height is usually given as:

$$H_{cr} = N_s \cdot \frac{c}{\gamma} \tag{1}$$

Where N_s is the *stability number* and γ is the soil self-weight and c is cohesion as stated before.

Statically admissible solution presents a lower bound solution for any stability problem. If an equilibrium distribution of stress can be found which balances applied load and nowhere violates the yield criterion, the soil mass will not fail or will be just at the point of failure [3].

Kinematic theorem states that the soil mass will collapse if there is any compatible pattern of plastic deformation (failure) of the soil mass for which virtual work of the external forces exceeds the virtual work done by internal forces in the soil [3]. Solving this problem analytically consists of calculating critical height for assumed failure patterns.

3. Lower bound solutions review

The stability problem of the vertical cut using static theorem was first examined by Drucker and Prager [4]. Soil mass was divided into three zones. Stress state was assumed for each of the zones and using Tresca's yield criterion it was calculated that the stability number is $N_s \geq 2$. Heyman [5] improved the lower bound solution by dividing the soil mass into seven zones, assuming stress state in each of the zones and solving the equilibrium equation using polar coordinates obtaining a closed-form solution where $N_s \geq 2.83$. Lower bound solution was further improved by Jong [6] $(N_s \geq 3.39)$, Pastor [7] $(N_s \geq 3.635)$ and Lyamin [8], being currently the highest lower bound solution $(N_s \geq 3.772)$ obtained using linear finite elements and non-linear programming.

4. Upper bound solutions review

Drucker and Prager also calculated the upper bound solution using limit analysis [4]. They assumed that the failure occurs along a straight line (plane), forming and angle of 45° with the horizontal axis and calculated that the stability number is $N_s \leq 4$. A better solution was obtained by Fellenius [9], by assuming a curved surface instead of plane $(N_s \leq 3.85)$. Assuming a rotational mechanism and using polar coordinates Chen [2] calculated the stability number to be $N_s \leq 3.831$. Bekaert [10] used a multi – rotation failure mechanism, obtaining the upper bound stability number $N_s \leq 3.793$. At the moment, lowest upper bound solution is presented by Pastor [11] using finite element method $\left(N_s \leq 3.786\right)$.

5. Upper bound solutions obtained in this paper

At the moment, upper and lower bound solutions are usually obtained numerically, using finite element method. In this paper, upper bound solutions were calculated analytically by assuming different failure patterns and defining critical height for each of the assumed failure patterns, by calculating the stability number. Critical height is calculated for soil mass in plane, assuming that the width of the observed soil mass is one meter.

Weight of the soil above the failure pattern represents external forces, while cohesion along the failure pattern represents internal force. Self-weight of the soil is γ and v is the virtual displacement along the failure pattern. H_{CR} is the critical height of the vertical cut, and χ is the coefficient that, when multiplied by H_{CR}, gives the value of the function for *y=0,* representing the horizontal length of the vertical cut at the soil surface (Figure 1).

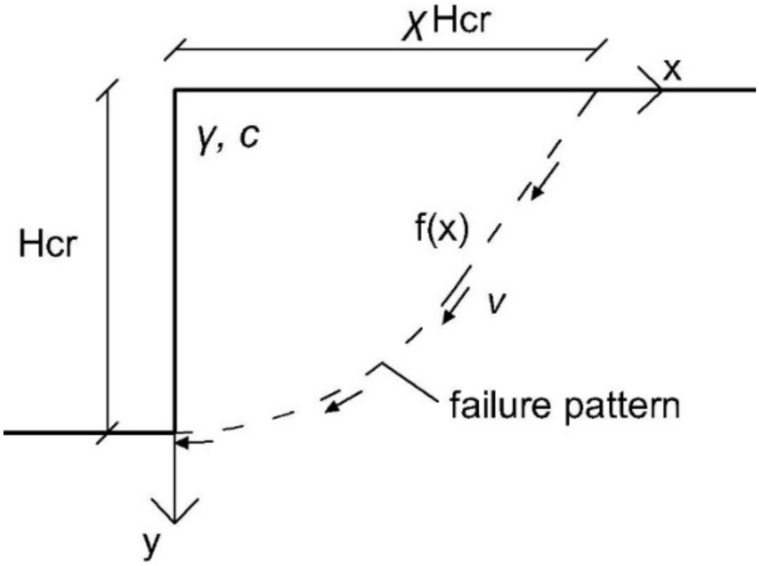

Figure 1: Failure pattern of a vertical cut

5.1 Failure pattern consisting of a circular arc and a straight line

As stated before, calculating the critical height of a vertical cut using kinematic theorem begins by assuming the failure pattern (failure mechanism). Here, it was assumed that failure pattern consists of a circular arc and a straight line. In order to calculate the work of internal and external forces, a two-dimensional coordinate system is introduced, whose orientation and position is shown in Figure 2. Position of points X_1, X_2, X_3 along the X-axis is defined using equation (2) and shown in Figure 2.

$$X_1 = 0; X_2 = H_{cr} \cdot \sin(\alpha); X_3 = {H_{cr}}/{\sin(\alpha)} \qquad (2)$$

The works of external forces and internal forces acting on the linear part of the failure pattern are marked as W_1 and D_1 respectively (Figure 3), and are defined in equations (3) and (4)

$$W_1 = \gamma \cdot (X_3 - X_1) \cdot \frac{X_2}{\tan(\alpha)} \cdot 0.5 \cdot v \cdot \sin(\alpha) \tag{3}$$

$$D_1 = c \cdot v \cdot \frac{(X_3 - X_2)}{\cos(\alpha)} \tag{4}$$

In order to define the work of the forces acting on the circular part of the failure pattern it is necessary to calculate the distance from the center of the circular arc to the rotation point (pole) of the soil above the circular arc X_1. Equations (5) and (6) define this distance - x_r .

$$r_t = \frac{2}{3} \cdot H_{cr} \cdot \frac{\sin(\alpha/2)}{\alpha/2} \tag{5}$$

$$x_r = r_t \cdot \sin(\alpha/2) \tag{6}$$

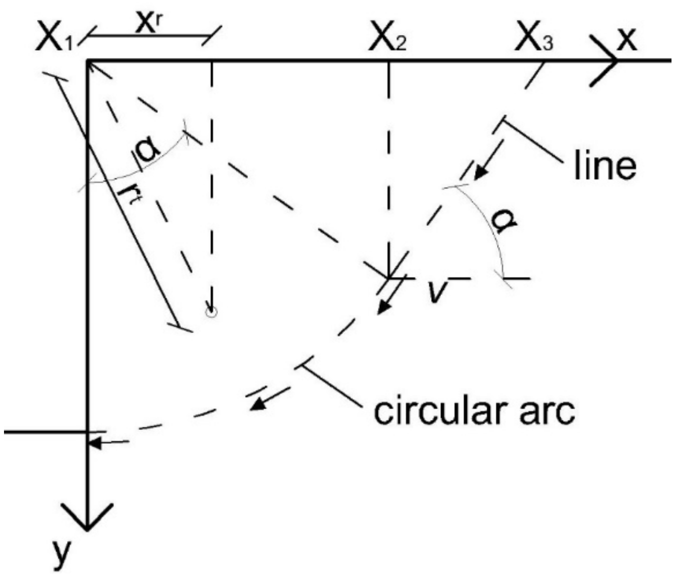

Figure 2: X-Y coordinate system

The works of external forces and internal forces acting on the circular part of the failure pattern are marked as W_2 and D_2 respectively (Figure 3), and are defined in equations (7) and (8).

$$W_2 = H_{cr}^2 \cdot \pi \cdot \gamma \cdot \frac{\alpha}{2 \cdot \pi} \cdot x_r \cdot \frac{v}{H_{cr}} \tag{7}$$

$$D_2 = 2 \cdot Hcr \cdot \pi \cdot \frac{\alpha}{2 \cdot \pi} \cdot c \cdot v \tag{8}$$

Total work of the external and internal forces is defined in equations (9),(10),(11) and (12).

$$W = W_1 + W_2 \tag{9}$$

$$D = D_1 + D_2 \tag{10}$$

$$W = \gamma \cdot H_{cr}^2 \cdot v \cdot \left(\frac{1}{2} \cdot \cos(\alpha) + \frac{2}{3} \cdot \left(\sin\left(\frac{\alpha}{2}\right) \right)^2 \right) \tag{11}$$

$$D = c \cdot v \cdot H_{cr} \cdot \left(\frac{1-(\sin(\alpha))^2}{\sin(\alpha)\cdot\cos(\alpha)} + \alpha \right) \tag{12}$$

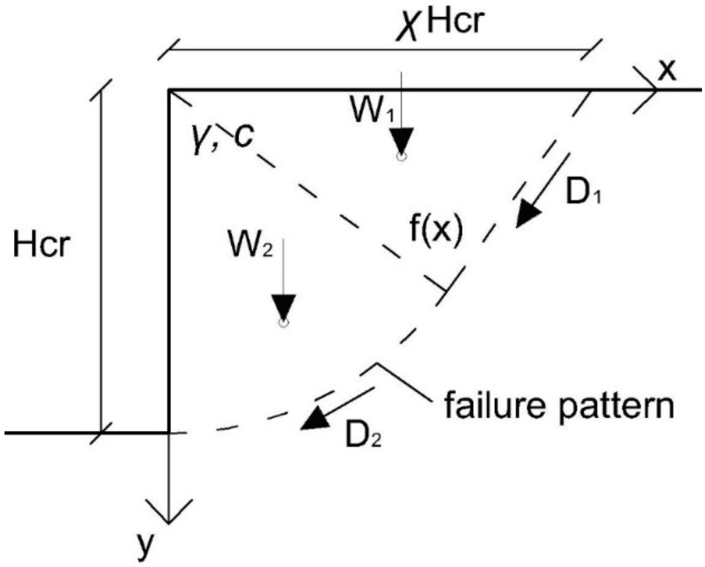

Figure 3: External and internal forces acting on the failure pattern

It was calculated that the lowest value of the stability number is obtained when $\alpha = 53.85^{\circ}$:

$$W = \gamma \cdot H_{cr}^2 \cdot v \cdot (0.2949 + 0.1367) = \gamma \cdot H_{cr}^2 \cdot v \cdot 0.43163 \tag{13}$$

$$D = c \cdot v \cdot H_{cr} \cdot (0.73 + 0.94) = c \cdot v \cdot H_{cr} \cdot 1.67 \tag{14}$$

$$D = W \rightarrow H_{cr} = 3.87 \cdot \frac{c}{\gamma} \tag{15}$$

5.2 Failure pattern defined with a function

Here, a new attempt for calculating the virtual work of external and internal forces is presented. When failure pattern is defined using a single function, work of external and internal forces is defined as:

$$W = \int_0^{\chi \cdot H_{cr}} f(x) \cdot \gamma \cdot \sin\left(arctg\left(\frac{df(x)}{dx} \right) \right) \cdot v \cdot dx \tag{16}$$

$$D = \int_0^{\chi \cdot H_{cr}} c \cdot v \cdot \sqrt{1 + \left| \frac{df(x)}{dx} \right|^2} \cdot dx \tag{17}$$

Where ***W*** defines the work of external forces (16), while ***D*** defines the work of internal forces (17). Stability number was obtained by solving equation (18):

$$D = W \tag{18}$$

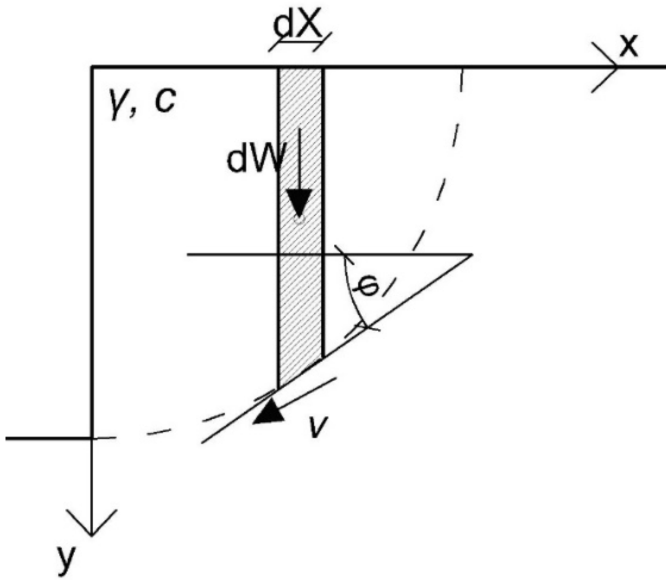

Figure 4: Failure pattern defined with a single function –differential part

Angle ***φ*** is defined as:

$$\varphi = arctg\left(\frac{df(x)}{dx}\right) \tag{19}$$

Function ***f(x)*** is the function that defines the shape of the failure pattern.

Work of the external and internal forces was calculated using integrals defined in equations (16) and (17). Program "Wolfram Alpha" was used to solve above-mentioned integrals. Then, a function was written in program "Matlab" that calculated stability numbers by solving the equation (18) for different values of constants defining the function and boundary coefficient ***χ***. Every function had three constants. Using boundary conditions, a link between constants was defined. So, for each function there was always one unknown constant and the boundary coefficient ***χ.***

5.2.1 Failure pattern defined with a sinus function

Firstly, a sinus function was used to define the failure pattern (20):

$$f(x) = a \cdot \sin(b \cdot x + c) \tag{20}$$

Constants ***a*** and ***b*** were expressed using constant ***c*** and boundary conditions (21)(22):

$$x = 0, y = -H_{cr} \rightarrow a = -\frac{H_{cr}}{\sin(c)} \tag{21}$$

$$y = 0, x = \chi \cdot H_{cr} \rightarrow b = -\frac{c}{\chi \cdot H_{cr}} \tag{22}$$

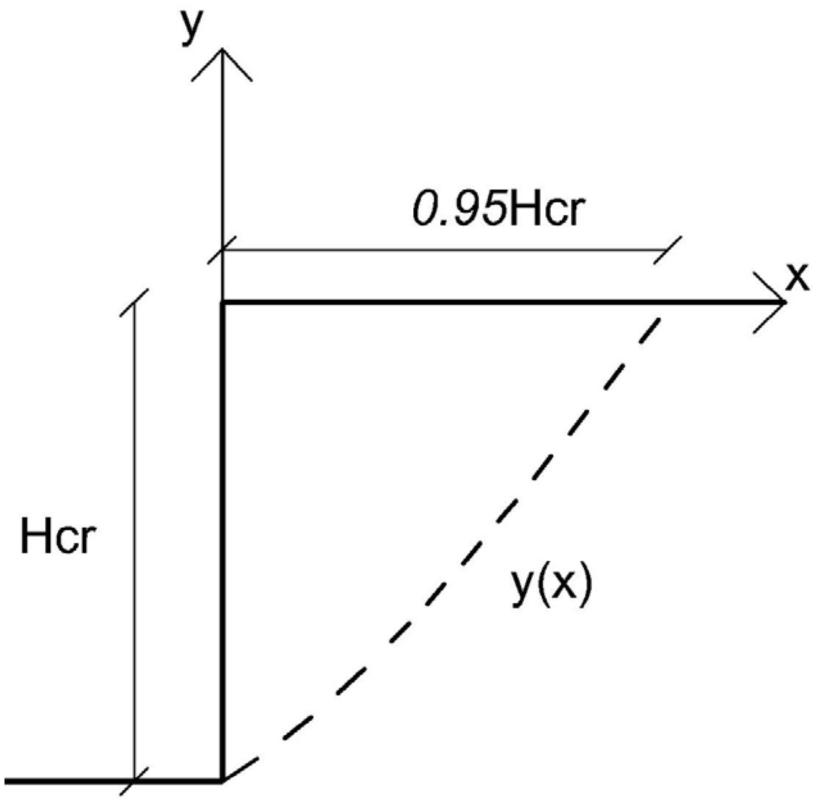

Figure 5: Failure pattern defined with a sinus function

The lowest value of the stability number obtained for the sinus function is $N_s = 3.884$, when $c = -1.088$ and $\chi = 0.95$. Sinus function for $c = -1.088$ and $\chi = 0.95$ is defined by equation (23) and shown in Figure 5.

$$y(x) = \frac{H_{cr}}{0.8857} \cdot \sin\left(\frac{1.145}{H_{cr}} \cdot x - 1.088 \right) \tag{23}$$

5.2.2 Failure pattern defined with a second degree polynomial

Second degree polynomial function defining the failure pattern is shown in the equation (24):

$$f(x) = a \cdot x^2 + b \cdot x + c \tag{24}$$

Constants ***a*** and ***c*** were expressed using constant ***b*** and boundary conditions (25)(26):

$$x = 0, y = H_{cr} \rightarrow c = H_{cr} \tag{25}$$

$$y = 0, x = \chi \cdot H_{cr} \rightarrow a = \frac{-1 - b \cdot \chi}{\chi^2 \cdot H_{cr}} \tag{26}$$

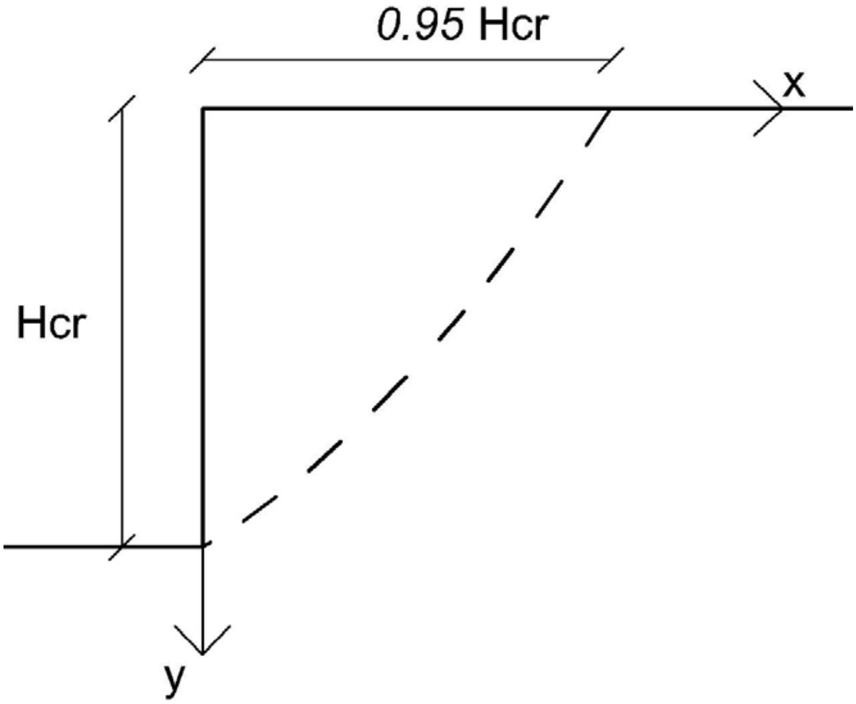

Figure 6: Failure pattern defined with a second degree polynomial

The lowest value of the stability number obtained for the second degree polynomial is $N_s = 3.856$ when $b = -0.6$ and $\chi = 0.95$. For these values of constants ***b*** and $\boldsymbol{\chi}$ second degree polynomial is defined by equation (27) and shown in Figure 6.

$$y(x) = \frac{-0.4765}{H_{cr}} \cdot x^2 - 0.6 \cdot x + H_{cr} \tag{27}$$

5.2.3 Failure pattern defined with an exponential function

Finally, it was assumed that the failure pattern is defined using an exponential function (28):

$$f(x) = a \cdot e^{b \cdot (-x)} + c \tag{28}$$

Constants ***b*** and ***c*** were expressed using constant ***a*** and boundary conditions (29)(30):

$$x = 0, y = H_{cr} \rightarrow c = H_{cr} - a \tag{29}$$

$$y = 0, x = -\chi \cdot H_{cr} \rightarrow b = \ln\left(\frac{a - H_{cr}}{a}\right) \cdot \frac{1}{\chi \cdot H_{cr}} \tag{30}$$

For the exponential function equation (16) is slightly different. Instead of $f(x), H_{cr} - f(x)$ is integrated. The lowest value of the stability number obtained for the exponential function is $N_s = 3.84$, where $a = 6.1$ and $\chi = 0.92$. Equation (31) defines the exponential function for $a = 6.1$ and $\chi = 0.92$. Figure 7 shows the exponential function defined in equation (31).

$$y(x)=6.1\cdot\exp\left(\ln\left(\frac{6.1-H_{cr}}{6.1}\right)\cdot\frac{1}{0.92\cdot H_{cr}}\cdot(-x)\right)+H_{cr}-6.1 \tag{31}$$

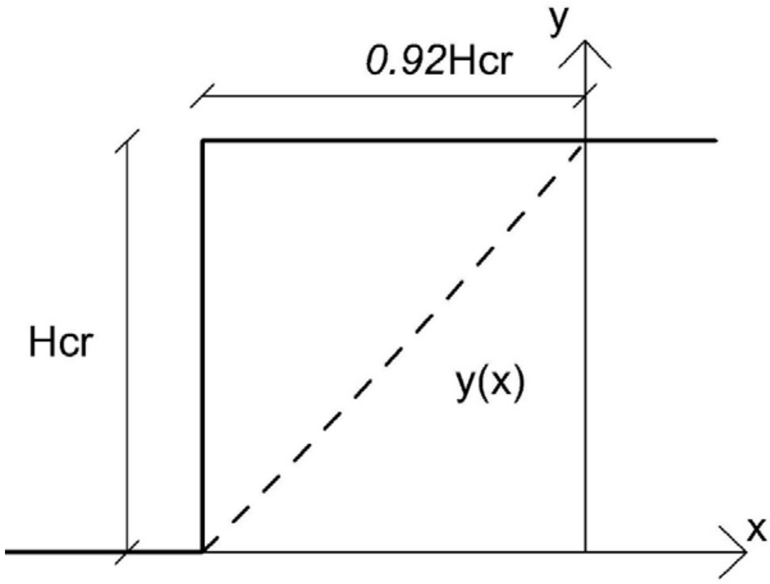

Figure 7: Failure pattern defined with an exponential function

6. Conclusion

In the present paper, the stability of a vertical cut was analyzed by calculating the critical height at which the cut becomes unstable under its self-weight. It was assumed that there is no internal friction in the soil (**φ**=0) and Coulomb's yield criterion was used for determining the failure of the soil mass. The theory behind plastic limit analysis theorems was briefly explained and a short review of existing upper bound and lower bound solutions was presented. Failure pattern consisting of a circular arc (curved surface) and straight line (plane) was assumed. Lowest value of the stability number for this type of failure pattern was calculated to be $N_s \leq 3.87$. Then, a new method for analytically calculating the virtual work of internal and external forces was presented. Integrals defined in equations (16) and (17) were used for calculating the stability number by solving the equation (18). Using programs "Matlab" and "Wolfram Alpha" stability number was calculated for three types of failure patterns defined by a function. First, stability number for a failure pattern defined using sinus function was calculated. The lowest value of the stability number for a failure pattern defined with a sinus function was $N_s = 3.884$. Then, a second degree polynomial function was used to define the failure pattern. Here, the lowest value of the stability number was $N_s = 3.856$. Finally, exponential function was used for defining the failure pattern. It was calculated that the lowest value of the stability number for a failure pattern defined by an exponential function was $N_s = 3.84$. Unfortunately, best upper bound solution obtained in this paper is still higher than the already existing upper bound solutions. But the method presented in this paper offers a relatively simple way of analytically calculating the stability number for various types of failure patterns defined by a function, which is very important for everyday engineering practice.

References:

[1] A. V. Lyamin and S. W. Sloan, “Upper bound limit analysis using linear finite elements and non-linear programming,” *Int. J. Numer. Anal. Methods Geomech.*, vol. 26, no. 2, pp. 181–216, Feb. 2002.

[2] W. F. Chen and C. Scawthorn, “Limit analysis and limit equilibrium solutions in soil mechanics, June 1968 (70-19),” 1968.

[3] Drucker D C and Prager W, “Soil mechanics and plastic analysis or limit design,” *Q. Appl. Math.*, vol. 10, p. 157–165.

[4] J. Heyman, “The stability of a vertical cut,” *Int. J. Mech. Sci.*, vol. 15, no. 10, pp. 845–854, 1973.

[5] G. D. J. D. Jong, “Improvement of the lowerbound solution for the vertical cut off in a cohesive, frictionless soil,” *Géotechnique*, vol. 28, no. 2, pp. 197–201, 1978.

[6] J. Pastor, “Analyse limite : ddtermination numdrique de solutions statiques completes. Application au talus vertical,” *J1 M6c Appl 2*, vol. n°2, pp. 167–196, 1978.

[7] A. V. Lyamin and S. W. Sloan, “Lower bound limit analysis using non-linear programming,” *Int. J. Numer. Methods Eng.*, vol. 55, no. 5, pp. 573–611, 2002.

[8] W. K. A. Fellenius, *Erdstatische Berechnungen mit Reibung und Koh�sion (Adh�sion) und unter Annahme kreiszylindrischer Gleitfl�chen,*. Berlin: W. Ernst & Sohn, 1927.

[9] W.-F. Chen, *Limit analysis and soil plasticity*. Elsevier, 2013.

[10] A. Bekaert, “Improvement of the kinematic bound for the stability of a vertical cut-off,” *Mech. Res. Commun.*, vol. 22, no. 6, pp. 533–540, Nov. 1995.

[11] J. Pastor, T.-H. Thai, and P. Francescato, “New bounds for the height limit of a vertical slope,” *Int. J. Numer. Anal. Methods Geomech.*, vol. 24, no. 2, pp. 165–182, 2000.