

Weighted quasisymmetric enumerator for generalized permutohedra

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Mathematics Subject Classifications: 52B40, 52B05, 16T05

Abstract

We introduce a weighted quasisymmetric enumerator function associated to generalized permutohedra. It refines the Billera, Jia and Reiner quasisymmetric function which also includes the Stanley chromatic symmetric function. Beside that it carries information of face numbers of generalized permutohedra. We consider more systematically the cases of nestohedra and matroid base polytopes.

Keywords: generalized permutohedron, quasisymmetric function, matroid, matroid base polytope, combinatorial Hopf algebra, f-polynomial

1 Introduction

For a generalized permutohedron Q there is a quasisymmetric enumerator function $F(Q)$ introduced by Billera, Jia and Reiner in [4]. It enumerates positive integer lattice points $\omega = (\omega_1, \dots, \omega_n) \in \mathbb{Z}_+^n$ which are Q -generic. It

means that the weight function $\omega^* : Q \rightarrow \mathbb{R}$ defined by $\omega^*(x) = \langle \omega, x \rangle$ has its maximum at a unique vertex v of Q . That is

$$F(Q) = \sum_{\omega \text{ } Q\text{-generic}} \mathbf{x}_\omega, \quad (1)$$

where $\mathbf{x}_\omega = x_{\omega_1} x_{\omega_2} \cdots x_{\omega_n}$. The generalized permutohedra are introduced and extensively studied by Postnikov [16] and Postnikov, Reiner and Williams [17]. They are deformations of the standard permutohedra obtained by moving facets in normal directions. The faces of the standard permutohedron Pe^{n-1} are labelled by flags \mathcal{F} of subsets of the set $[n]$. A part of the reach combinatorial structure of a generalized permutohedron Q is a certain statistic rk_Q on the face lattice of the standard permutohedron Pe^{n-1} which we call the Q -rank. The Q -rank of a face \mathcal{F} of Pe^{n-1} is the dimension of the face of Q in which \mathcal{F} is deformed. The normal fan of the standard permutohedron $\Sigma_{Pe^{n-1}}$ is the braid arrangement fan. The space \mathbb{R}^n is decomposed by braid cones $\sigma_{\mathcal{F}}$. Each positive integer vector $\omega \in \mathbb{Z}_+^n$ determines a unique flag \mathcal{F}_ω such that ω lies in the relative interior of the braid cone $\sigma_{\mathcal{F}_\omega}$. We define the following weighted enumerator function associated to a generalized permutohedron Q

$$F_q(Q) = \sum_{\omega \in \mathbb{Z}_+^n} q^{\text{rk}_Q(\mathcal{F}_\omega)} \mathbf{x}_\omega. \quad (2)$$

The Q -rank of a face \mathcal{F} is zero if it is deformed into a vertex of Q . Henceforth $F_q(Q)$ specializes at $q = 0$ to the Billera, Jia and Reiner quasisymmetric enumerator function $F_0(Q) = F(Q)$.

On the other hand, when a class of generalized permutohedra is specified we obtain the well known combinatorial enumerators. The case of graphical zonotopes $Q = Z_\Gamma$ is studied in [12]. The quasisymmetric function $F_q(Z_\Gamma)$ is a q -refinement of the Stanley chromatic symmetric function X_Γ of graphs

$$F_0(Z_\Gamma) = X_\Gamma.$$

The case of nestohedra is studied in [11] for $q = 0$ and for a q -analog in [13]. For a subclass of graph-associahedra $Q = P_\Gamma$ the enumerator $F(P_\Gamma)$ produces a new quasisymmetric invariant of a graph Γ with a nice behavior.

The both cases of graphical zonotopes and nestohedra have in common that corresponding enumerators $F(Q)$ coincide with universal morphisms from certain combinatorial Hopf algebras to quasisymmetric functions. In the

case of graphical zonotopes it is the chromatic Hopf algebra of graphs and in the case of nestohedra it is the non-cocommutative Hopf algebra on building sets and its Hopf subalgebra of graphs. Billera, Jia and Reiner applied their enumerator $F(Q)$ in the case of matroid base polytopes $Q = P_M$. The invariant $F(P_M)$ also comes from a Hopf algebra, in this case of matroids which was firstly introduced by Schmitt [18].

A remarkable and unifying approach to Hopf monoid structures constructed on combinatorial objects that provide generalized permutohedra have been developed in the recently published paper by M. Aguiar and F. Ardila [1]. Combining with the universality of quasisymmetric functions in the category of combinatorial Hopf algebras [2] shows the naturality of the invariant $F_q(Q)$ in enumerative and algebraic combinatorics.

The paper is organized as follows. In section 2 we review the necessary facts about combinatorics of standard and generalized permutohedra. In section 3 we review the basic facts about quasisymmetric functions. In section 4 we introduce the quasisymmetric function $F_q(Q)$ and show that it contains the information about f -vectors of generalized permutohedra. It may be regarded as a far-reaching illumination of the Stanley (-1) -color theorem for numbers of acyclic orientations of a graph. The cases of nestohedra and graph-associahedra are considered in sections 5 and 6. In the rest of the paper the case of matroid base polytopes is considered more thoroughly. In section 7 we review some basic facts about combinatorics of matroid base polytopes and introduce the combinatorial Hopf algebra of matroids and its q -analog. Some calculation for uniform matroids is presented. Finally in section 8 some properties of the weighted quasisymmetric enumerator function of matroids are derived.

2 Generalized permutohedra

The symmetric group S_n acts on the space \mathbb{R}^n by permuting the coordinates. Recall that a $(n - 1)$ -dimensional *permutohedron* Pe^{n-1} is the convex hull of the orbit of a point with increasing coordinates $x_1 < \dots < x_n$

$$Pe^{n-1} = \text{Conv}\{(x_{\omega(1)}, x_{\omega(2)} \dots, x_{\omega(n)}) \mid \omega \in S_n\}.$$

The *braid arrangement* is the arrangement of hyperplanes $\{x_i = x_j\}_{1 \leq i < j \leq n}$ in the space \mathbb{R}^n . The regions of the braid arrangement, called

Weyl chambers are labelled by permutations $\omega \in S_n$

$$C_\omega := \{x_{\omega(1)} \leq x_{\omega(2)} \leq \dots \leq x_{\omega(n)}\}.$$

The corresponding *braid arrangement fan* is the normal fan $\Sigma_{Pe^{n-1}}$ of the permutohedron Pe^{n-1} . The cones of the braid arrangement fan are called *braid cones*.

A flag of the length $|\mathcal{F}| = k$ on the set $[n] = \{1, \dots, n\}$ is a chain of subsets

$$\mathcal{F} : \emptyset =: F_0 \subset F_1 \subset \dots \subset F_{k-1} \subset F_k \subset F_{k+1} := [n].$$

The type of a flag \mathcal{F} is the following $(k+1)$ -tuple of integers

$$\text{type}(\mathcal{F}) = (|F_1| - |F_0|, |F_2| - |F_1|, \dots, |F_{k+1}| - |F_k|). \quad (3)$$

The set of flags is ordered by refinements. We write $\mathcal{G} \preceq \mathcal{F}$ if \mathcal{F} refines \mathcal{G} . There is an obvious order reversing one-to-one correspondence between the face lattice of the permutohedron Pe^{n-1} and the lattice of flags of $[n]$. With no abuse of notation we denote a face of the permutohedron Pe^{n-1} by the corresponding flag \mathcal{F} . Then $\mathcal{G} \preceq \mathcal{F}$ if and only if $\mathcal{F} \subseteq \mathcal{G}$ as faces of Pe^{n-1} . By this convention we have that $\dim \mathcal{F} = n - |\mathcal{F}| - 1$. For example, the facets correspond to the flags $\emptyset \subset A \subset [n]$, while the vertices correspond to maximal flags.

The braid cone $\sigma_{\mathcal{F}}$ at the face \mathcal{F} is determined by the coordinates relations

$$\begin{cases} x_p = x_q & \text{if } p, q \in F_{i+1} \setminus F_i \text{ for some } i = 0, \dots, k, \\ x_p \leq x_q & \text{if } p \in F_i \setminus F_{i-1} \text{ and } q \in F_{i+1} \setminus F_i \text{ for some } i = 1, \dots, k. \end{cases} \quad (4)$$

Note that $\dim(\sigma_{\mathcal{F}}) = |\mathcal{F}|$ and the relative interior $\sigma_{\mathcal{F}}^\circ$, given by strict inequalities in the second condition above, is homeomorphic to $\mathbb{R}^{|\mathcal{F}|}$. Conversely, the flag \mathcal{F} can be reconstructed from the braid cone $\sigma_{\mathcal{F}}$ by setting $p \in F_{i+1} \setminus F_i$ and $q \in F_{j+1} \setminus F_j$ for some $0 \leq i < j \leq k$ whenever $x_p < x_q$ for all points in the relative interior $\sigma_{\mathcal{F}}^\circ$.

Definition 2.1. A convex polytope Q is an $(n-1)$ -dimensional *generalized permutohedron* if its normal fan Σ_Q is coarser than the braid arrangement fan $\Sigma_{Pe^{n-1}}$.

Generalized permutohedra are equivalently characterized as deformations of the standard permutohedron Pe^{n-1} by moving its vertices with keeping

directions of edges. Thus any edge of a generalized permutohedron Q lies in the direction of some $e_i - e_j$, where $e_i, i = 1, \dots, n$ are the standard basis vectors in \mathbb{R}^n . For equivalent descriptions of generalized permutohedra see [17].

Definition 2.2. For an $(n - 1)$ -dimensional generalized permutohedron Q there is a map of face lattices

$$\pi_Q : L(Pe^{n-1}) \rightarrow L(Q)$$

determined by $\pi_Q(\mathcal{F}) = G$ if and only if the relative interior of the braid cone $\sigma_{\mathcal{F}}^{\circ}$ is contained in the relative interior σ_G° of the normal cone at the face G of Q .

Proposition 2.3.

$$\sum_{\mathcal{F}:\pi_Q(\mathcal{F})=G} (-1)^{|\mathcal{F}|} = (-1)^{n-\dim(G)-1}.$$

Proof. Consider the collection of flags $\pi_Q^{-1}(G) = \{\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_t\}$. We have $\sigma_G^{\circ} = \bigcup_{i=1}^t \sigma_{\mathcal{F}_i}^{\circ}$ and σ_G° is homeomorphic to the k -dimensional open cell, where $k = n - \dim G - 1$. Let $f_i, i = 0, 1, \dots, k$ be the numbers of i -dimensional cones such that $\sigma_{\mathcal{F}}^{\circ} \subset \sigma_G^{\circ}$. By inclusion-exclusion principle we have

$$f_k - f_{k-1} + \dots + (-1)^k f_0 = 1.$$

□

3 Quasisymmetric functions

In this section we review the basic facts about combinatorial Hopf algebras and quasisymmetric functions. The notion of combinatorial Hopf algebra, originated in the work of Aguiar, Bergeron and Sottile [2], gives a natural algebraic framework of enumerative combinatorics. The extensive survey of Hopf algebra theory in combinatorics may be found in [10].

A combinatorial Hopf algebra is a graded, connected Hopf algebra \mathcal{H} equipped with a multiplicative linear functional $\zeta : \mathcal{H} \rightarrow \mathbf{k}$ to the ground field. We describe the combinatorial Hopf algebra $QSym$ of quasisymmetric functions.

A quasisymmetric function $F = F(\mathbf{x})$ is a formal power series of bounded degree in the countable ordered set of variables $\mathbf{x} = (x_1, x_2, \dots)$ such that coefficients by monomials with the same list of ordered exponents are equal. This condition produces the natural linear basis for the algebra $QSym$ consisting of monomial quasisymmetric functions

$$M_\alpha = \sum_{i_1 < i_2 < \dots < i_k} x_{i_1}^{a_1} x_{i_2}^{a_2} \dots x_{i_k}^{a_k},$$

indexed by finite ordered sets of integers $\alpha = (a_1, a_2, \dots, a_k)$ called compositions of $|\alpha| = a_1 + a_2 + \dots + a_k$ of the length $k(\alpha) = k$. The coproduct, defined on monomial basis by

$$\Delta(M_\alpha) = \sum_{\beta\gamma=\alpha} M_\beta \otimes M_\gamma,$$

where $\beta\gamma$ is the concatenation of compositions, turns $QSym$ into a graded, connected Hopf algebra.

The principal specialization $\mathbf{ps} : QSym \rightarrow \mathbf{k}[m]$ assigns to a quasisymmetric function F a polynomial in m by evaluation map

$$\mathbf{ps}(F)(m) = F|_{x_1=\dots=x_m=1, x_{m+1}=\dots=0}.$$

The canonical character on quasisymmetric functions $\zeta_Q : QSym \rightarrow \mathbf{k}$ is defined by

$$\zeta_Q(F) = \mathbf{ps}(F)(1).$$

It is easy to see that on monomial basis we have

$$\mathbf{ps}(M_\alpha)(m) = \binom{m}{k(\alpha)}$$

and specially

$$\mathbf{ps}(M_\alpha)(-1) = (-1)^{k(\alpha)}. \tag{5}$$

The following theorem is fundamental in applications and explains the ubiquity of quasisymmetric functions as enumerator functions in combinatorics.

Theorem 3.1 ([2], Theorem 4.1). *For a combinatorial Hopf algebra (\mathcal{H}, ζ) there is a unique morphism of graded Hopf algebras*

$$\Psi : \mathcal{H} \rightarrow QSym$$

such that $\Psi \circ \zeta_Q = \zeta$. For a homogeneous element h of degree n the coefficients $\zeta_\alpha(h)$, $\alpha = (a_1, \dots, a_k)$ of $\Psi(h)$ in monomial basis of $QSym$ are given by

$$\zeta_\alpha(h) = \zeta^{\otimes k} \circ (p_{a_1} \otimes p_{a_2} \otimes \dots \otimes p_{a_k}) \circ \Delta^{(k-1)}(h),$$

where p_i is the projection on the i -th homogeneous component and $\Delta^{(k-1)}$ is the $(k-1)$ -fold coproduct map of \mathcal{H} .

4 Weighted quasisymmetric enumerator $F_q(Q)$

A generalized permutohedron Q comes with a natural map between face lattices $\pi_Q : L(Pe^{n-1}) \rightarrow L(Q)$ given by Definition 2.2 with $\pi_Q(\mathcal{F}) = G$ if and only if $\sigma_{\mathcal{F}}^\circ \subset \sigma_G^\circ$. This map produces a natural statistic of faces of the standard permutohedron Pe^{n-1} .

Definition 4.1. For a generalized permutohedron Q the Q -rank is a map on the face lattice of the standard permutohedron $\text{rk}_Q : L(Pe^{n-1}) \rightarrow \{0, 1, \dots, n-1\}$ given by

$$\text{rk}_Q(\mathcal{F}) = \dim(\pi_Q(\mathcal{F})).$$

Let $\omega = (\omega_1, \omega_2, \dots, \omega_n) \in \mathbb{Z}_+^n$ be an integer lattice vector with positive entries. It defines the *weight function* $\omega^* : Q \rightarrow \mathbb{R}$ on Q by $\omega^*(x) = \langle \omega, x \rangle$, where \langle, \rangle is the standard scalar product in \mathbb{R}^n . Note that Q lies in a hyperplane whose normal vector is $(1, \dots, 1)$. The weight function ω^* is maximized along a unique face G_ω of Q which is determined by the condition that the vector ω lies in the relative interior of its normal cone $\omega \in \sigma_{G_\omega}^\circ$. A weight function ω^* is called Q -generic if G_ω is a vertex of Q .

If $Q = Pe^{n-1}$ is the standard permutohedron each $\omega \in \mathbb{Z}_+^n$ determines a unique flag \mathcal{F}_ω by the condition $\mathcal{F} = \mathcal{F}_\omega$ if and only if the integer vector ω lies in the relative interior of the corresponding braid cone $\omega \in \sigma_{\mathcal{F}}^\circ$.

Definition 4.2. For a generalized permutohedron Q let $F_q(Q)$ be a weighted enumerator of positive integer vectors

$$F_q(Q) = \sum_{\omega \in \mathbb{Z}_+^n} q^{\text{rk}_Q(\mathcal{F}_\omega)} \mathbf{x}_\omega.$$

We expand the enumerator function $F_q(Q)$ in the monomial bases of quasisymmetric functions.

Definition 4.3. For a flag \mathcal{F} of subsets of $[n]$ let $M_{\mathcal{F}}$ be the enumerator of positive integer vectors in relative interior of the corresponding braid cone

$$M_{\mathcal{F}} = \sum_{\omega \in \mathbb{Z}_+^n \cap \sigma_{\mathcal{F}}^\circ} \mathbf{x}_\omega.$$

An enumerator $M_{\mathcal{F}}$ is exactly the monomial quasisymmetric function depending only on the type of \mathcal{F} , given by (3)

$$M_{\mathcal{F}} = M_{\text{type}(\mathcal{F})}.$$

We obtain the expansion of $F_q(Q)$ according to the face lattice of the standard permutohedron

$$F_q(Q) = \sum_{\mathcal{F} \in L(Pe^{n-1})} q^{\text{rk}_Q(\mathcal{F})} M_{\mathcal{F}}. \quad (6)$$

In the monomial basis $F_q(Q)$ has the expansion of the form

$$F_q(Q) = \sum_{\alpha \models n} p_\alpha(q) M_\alpha$$

where $p_\alpha(q)$ are polynomials in q indexed by compositions of n . For a composition α the polynomial $p_\alpha(q)$ is given by

$$p_\alpha(q) = \sum_{\mathcal{F}: \text{type}(\mathcal{F})=\alpha} q^{\text{rk}_Q(\mathcal{F})}.$$

By definition 2.2 of the map π_Q we have

$$\sum_{\omega \in \sigma_G^\circ} \mathbf{x}_\omega = \sum_{\mathcal{F}: \pi_Q(\mathcal{F})=G} M_{\mathcal{F}},$$

which gives the expansion of $F_q(Q)$ in terms of the face lattice of Q

$$F_q(Q) = \sum_{G \in L(Q)} q^{\dim(G)} \sum_{\mathcal{F}: \pi_Q(\mathcal{F})=G} M_{\mathcal{F}}. \quad (7)$$

From this we easily derive that the enumerator F_q contains the information about f -vectors of generalized permutohedra. The f -vector of a convex $(n-1)$ -dimensional polytope P is the integer vector $f = (f_0, f_1, \dots, f_{n-1})$, where f_i is the number of i -dimensional faces of P . It is codified by the f -polynomial

$$f(P, q) = f_0 + f_1q + f_2q^2 + \dots + f_{n-1}q^{n-1}.$$

Theorem 4.4. *The f -polynomial $f(Q, q)$ of an $(n-1)$ -dimensional generalized permutohedron Q is determined by the principal specialization*

$$f(Q, q) = (-1)^n \mathbf{ps}(F_{-q}(Q))(-1).$$

Proof. By formula (7) and the fact $\mathbf{ps}(M_{\mathcal{F}})(-1) = (-1)^{|\mathcal{F}|+1}$ implied by (5) the principal specialization of $F_q(Q)$ gives

$$(-1)^n \mathbf{ps}(F_{-q}(Q))(-1) = \sum_{G \in L(Q)} q^{\dim(G)} \sum_{\mathcal{F}: \pi_Q(\mathcal{F})=G} (-1)^{|\mathcal{F}|+1+n+\dim(G)}.$$

The inner sum is equal to 1 by Proposition 2.3, which completes the proof. \square

4.1 Action of the antipode on $F_q(Q)$

In this subsection we determine how the antipode S of the Hopf algebra of quasisymmetric functions $QSym$ acts on the weighted quasisymmetric enumerator function $F_q(Q)$.

Define the opposite flag \mathcal{F}^{op} to a flag $\mathcal{F} : \emptyset =: F_0 \subset F_1 \subset \dots \subset F_k \subset F_{k+1} := [n]$ by

$$\mathcal{F}^{op} : \emptyset \subset [n] \setminus F_k \subset \dots \subset [n] \setminus F_1 \subset [n].$$

The corresponding braid cones are related by $\sigma_{\mathcal{F}^{op}} = -\sigma_{\mathcal{F}}$. For a composition $\alpha = (a_1, \dots, a_k)$ the reverse composition is $\text{rev}(\alpha) = (a_k, \dots, a_1)$. We have

$$\text{type}(\mathcal{F}^{op}) = \text{rev}(\text{type}(\mathcal{F})). \quad (8)$$

Recall that flags are ordered by refinements. The following lemma gives a particularly nice geometric meaning of the formula for the antipode S , see [10, Theorem 5.11] and reference within.

Lemma 4.5. *The antipode S on the monomial quasisymmetric function $M_{\mathcal{F}}$ associated to a flag \mathcal{F} acts by*

$$S(M_{\mathcal{F}}) = (-1)^{|\mathcal{F}|+1} \sum_{\mathcal{G} \preceq \mathcal{F}^{op}} M_{\mathcal{G}}.$$

This allows us to interpret $S(M_{\mathcal{F}})$ as the enumerator function of integer lattice points lying in the opposite braid cone $\sigma_{\mathcal{F}^{op}}$

$$(-1)^{|\mathcal{F}|+1} S(M_{\mathcal{F}}) = \sum_{\omega \in \mathbb{Z}_+^n \cap \sigma_{\mathcal{F}^{op}}} \mathbf{x}_{\omega}.$$

Let π_Q be the map associated to a generalized permutohedron Q by definition 2.2. We say that the face $\pi_Q(\mathcal{F}^{op})$ is opposite to a flag \mathcal{F} . The following theorem describes the action of the antipode on the weighted quasisymmetric enumerator function $F_q(Q)$.

Theorem 4.6. *Given a generalized permutohedron Q of dimension $n-1$ the antipode S acts on the weighted quasisymmetric enumerator function $F_q(Q)$ by*

$$S(F_q(Q)) = (-1)^n \sum_{\mathcal{G}} f(\pi_Q(\mathcal{G}^{op}), -q) M_{\mathcal{G}},$$

where the sum is over all flags \mathcal{G} of the set $[n]$ and $f(\pi_Q(\mathcal{G}^{op}), q)$ is the f -polynomial of the face $\pi_Q(\mathcal{G}^{op})$ opposite to a flag \mathcal{G} .

Proof. By the expansion (6) and lemma 4.5 we have

$$S(F_q(Q)) = \sum_{\mathcal{F}} q^{\text{rk}_Q(\mathcal{F})} S(M_{\mathcal{F}}) = \sum_{\mathcal{F}} q^{\text{rk}_Q(\mathcal{F})} (-1)^{|\mathcal{F}|+1} \sum_{\mathcal{G} \preceq \mathcal{F}^{op}} M_{\mathcal{G}},$$

which gives

$$S(F_q(Q)) = \sum_{\mathcal{G}} M_{\mathcal{G}} \sum_{\mathcal{F}: \mathcal{G}^{op} \preceq \mathcal{F}} (-1)^{|\mathcal{F}|+1} q^{\text{rk}_Q(\mathcal{F})}.$$

It remains to show that

$$\sum_{\mathcal{F}: \mathcal{G}^{op} \preceq \mathcal{F}} (-1)^{|\mathcal{F}|+1} q^{\text{rk}_Q(\mathcal{F})} = (-1)^n f(\pi_Q(\mathcal{G}^{op}), -q),$$

i.e.

$$f(\pi_Q(\mathcal{G}^{op}), q) = \sum_{\mathcal{F}: \mathcal{G}^{op} \preceq \mathcal{F}} (-1)^{|\mathcal{F}|+1+n+\text{rk}_Q(\mathcal{F})} q^{\text{rk}_Q(\mathcal{F})},$$

which is a consequence of Definition 4.1 and Proposition 2.3. \square

Theorem 4.6 generalizes Theorem 4.4. To see this note that $\pi_Q(\emptyset \subset [n]) = Q$ and by Theorem 4.6 the coefficient in $S(F_q(Q))$ by monomial quasisymmetric function $M_n = M_{\emptyset \subset [n]}$ is equal to $(-1)^n f(Q, -q)$. This coefficient can be extracted from $S(F_q(Q))$ by composing with the canonical character ζ_Q on quasisymmetric functions. Theorem 4.4 then follows from the fact

$$\zeta_Q \circ S(F) = \mathbf{ps}(F)(-1), \quad (9)$$

which can be easily seen to hold for monomial bases and consequently for each quasisymmetric function F .

Corollary 4.7. *For a generalized permutohedron Q the following identity holds*

$$\mathbf{ps}(S(F_q(Q)))(-1) = q^{\dim(Q)}.$$

Proof. It follows from the equation (9) and the fact that the antipode S of $QSym$ is of order two $S^2 = \text{Id}$. \square

4.2 Specializations of the enumerator $F_q(Q)$

We review some specializations of the quasisymmetric enumerator function $F_q(Q)$. By taking $q = 0$ it is specified to the Billera, Jia and Reiner quasisymmetric function $F(Q)$ which is the enumerator of Q -generic weight vectors

$$F(Q) = \sum_{\omega \in \mathbb{Z}_+^n : \text{rk}(\mathcal{F}_\omega) = 0} \mathbf{x}_\omega.$$

Theorem 4.4 may be regarded as the generalization of the formula for number of vertices f_0 of a generalized permutohedron Q , see [4, Theorem 9.2]

$$f_0 = (-1)^n \mathbf{ps}(F(Q))(-1).$$

Note that $F_q(Q)$ degenerates at $q = 1$ since $F_1(Q) = \sum_{\omega \in \mathbb{Z}_+^n} \mathbf{x}_\omega = (M_1)^n$. Theorem 4.4 then gives

$$f(Q, -1) = f_0 - f_1 + \cdots + (-1)^{n-1} f_{n-1} = (-1)^n \mathbf{ps}(M_1^n)(-1) = 1,$$

which is Euler characteristic of Q .

For particular classes of generalized permutohedra the enumerator F_q specializes to known quasisymmetric invariants as it is announced in introduction. We briefly review the case of graphical zonotopes and more thoroughly the cases of nestohedra and matroid base polytopes in subsequent sections.

Graphical zonotopes The vertices of a graphical zonotope Z_Γ are in one-to-one correspondence with regions of the corresponding graphical arrangement. The integer points $\omega \in \mathbb{Z}_+^n$ can be interpreted as graph colorings by labelling vertices of a graph Γ . The normal cone at a vertex of Z_Γ is a region of the corresponding graphical arrangement. So ω determines a proper coloring of the graph Γ if and only if $\text{rk}_{Z_\Gamma}(\mathcal{F}_\omega) = 0$. Thus the enumerators of proper colorings and Z_Γ -generic integer vectors coincide, i.e. $F_0(Z_\Gamma) = X_\Gamma$, where X_Γ is the Stanley chromatic symmetric function (introduced in [19]).

The main argument in [12] of the proof of Theorem 4.4 for graphical zonotopes was based on the Humpert and Martin cancelation-free formula for the antipode of the chromatic Hopf algebra of graphs [14].

5 Nestohedra

We refer the reader to [17] for definitions and main properties of nestohedra. The nestohedra are a class of simple polytopes described by the notion of building sets. A collection of subsets B of a finite ground set V is a building set if

- ◊ $\{i\} \in B$ for all $i \in V$ and
- ◊ if $I, J \in B$ and $I \cap J \neq \emptyset$ then $I \cup J \in B$.

A building set B is connected if $V \in B$. Let $\Delta^{n-1} = \text{Conv}\{e_1, \dots, e_n\}$ be the standard simplex in \mathbb{R}^n . The nestohedron associated to a building set B on the ground set $[n]$ is the Minkowsky sum of simplexes $P_B =$

$\sum_{I \in B} \text{Conv}\{e_i \mid i \in I\}$. Enumerate faces of Δ^{n-1} by subsets of $[n]$ in a way that the face poset of Δ^{n-1} is isomorphic to the reverse Boolean lattice on $[n]$. For a connected building set B the nestohedron P_B is realized by successive truncations over faces of Δ^{n-1} encoded by a building set B in any nondecreasing sequence of dimensions of faces. Thus facets of P_B are labelled by elements $I \in B \setminus \{[n]\}$. Recall that a truncation of a convex polytope P over a face $F \subset P$ is the polytope $P \setminus F$ obtained by cutting P with a hyperplane H_F which divides vertices in F and vertices not in F in separated halfspaces. For a disconnected building set B the associated nestohedron P_B is the product of nestohedra corresponding to components of B .

The following condition describes the face poset of a nestohedron P_B corresponding to a connected building set B , see [7], Theorem 3.14 and [16], Theorem 7.4. The intersection $F_{I_1} \cap \dots \cap F_{I_k}, k \geq 2$ of facets corresponding to a subcollection $N = \{I_1, \dots, I_k\} \subset B \setminus \{[n]\}$ is a nonempty face of P_B if and only if

(N1) $I_i \subset I_j$ or $I_j \subset I_i$ or $I_i \cap I_j = \emptyset$ for any $1 \leq i < j \leq k$,

(N2) $I_{j_1} \cup \dots \cup I_{j_p} \notin B$ for any pairwise disjoint sets I_{j_1}, \dots, I_{j_p} .

Subcollections that satisfy conditions (N1) and (N2) form a simplicial complex, called the *nested set complex*, whose face poset is opposite to the face poset of P_B .

5.1 Hopf algebra \mathcal{B}

Two building sets B_1 and B_2 are isomorphic if there is a bijection of their sets of vertices $f : V_1 \rightarrow V_2$ such that $I \in B_1$ if and only if $f(I) \in B_2$. The addition of building sets B_1 and B_2 on disjoint ground sets V_1 and V_2 is the building set $B_1 \sqcup B_2 = \{I \subset V_1 \sqcup V_2 \mid I \in B_1 \text{ or } I \in B_2\}$. For a building set B on V and a subset $S \subset V$ the restriction on S and the contraction of S from B are defined by $B|_S = \{I \subset S \mid I \in B\}$ and $B/S = \{I \subset V \setminus S \mid I \in B \text{ or } I \cup S' \in B \text{ for some } S' \subset S\}$. The building sets obtained from B by restrictions and contractions are its minors.

The following combinatorial Hopf algebra of building set is considered in [11]. The set of all isomorphism classes of finite building sets linearly generates the vector space \mathcal{B} over a field \mathbf{k} . The space \mathcal{B} is a graded, commutative and non-cocommutative Hopf algebra with the multiplication and

the comultiplication

$$[B_1] \cdot [B_2] = [B_1 \sqcup B_2] \quad \text{and} \quad \Delta([B]) = \sum_{S \subset V} [B|_S] \otimes [B/S].$$

The grading $\text{gr}([B])$ is given by the cardinality of the ground set of B . A building set B is connected if $[B]$ is irreducible, i.e. it is not represented by an addition of two building sets. Denote by $c(B)$ the number of connected components of B . Let $\zeta : \mathcal{B} \rightarrow \mathbf{k}$ be a multiplicative linear functional defined by $\zeta([B]) = 1$ if B is a discrete (consisting of singletons only) and $\zeta([B]) = 0$ otherwise. A unique morphism $\Psi : (\mathcal{B}, \zeta) \rightarrow (QSym, \zeta_Q)$ of combinatorial Hopf algebras is given in the monomial basis of quasismmetric functions by

$$\Psi([B]) = \sum_{\alpha = \text{gr}(B)} \zeta_\alpha(B) M_\alpha.$$

The coefficients $\zeta_\alpha(B)$ have an enumerative meaning. Let $\mathcal{L} : \emptyset \subset I_1 \subset \dots \subset I_k = V$ be a chain of subsets of the ground set $[n]$. Denote by $|\mathcal{L}| = k$ its length and by $\text{type}(\mathcal{L})$ its type which is a composition $\alpha = (i_1, \dots, i_k)$ such that for any $1 \leq j \leq k$ the set $I_j \setminus I_{j-1}$ has i_j elements. We say that \mathcal{L} is a splitting chain if all minors $B|_{I_j}/I_{j-1}$ are discrete. Then $\zeta_\alpha(B)$ is exactly the number of all splitting chains of B of a given type α . For a building set B on $[n]$ the following identity holds ([11, Theorem 4.5])

$$F(P_B) = \Psi([B]). \tag{10}$$

5.2 q -analog

We extend the basic field \mathbf{k} into the field of rational functions $\mathbf{k}(q)$ and define the character $\zeta_q : \mathcal{B} \rightarrow \mathbf{k}(q)$ with $\zeta_q([B]) = q^{\text{rk}(B)}$, where $\text{rk}(B) = \text{gr}(B) - c(B)$. Let $\Psi_q : (\mathcal{B}, \zeta_q) \rightarrow (QSym, \zeta_Q)$ be a unique morphism of combinatorial Hopf algebras over $\mathbf{k}(q)$.

Recall that a reflexive and transitive relation \preceq on $[n]$ is called a *preorder*. If in addition $p \preceq q$ or $q \preceq p$ for any $p, q \in [n]$ it is called a *weak order* on $[n]$. Weak orders correspond to set compositions of $[n]$. Each set composition determines a unique flag of subsets and vice-versa. We say that a flag of subsets $\mathcal{L} : \emptyset = I_0 \subset I_1 \subset \dots \subset I_k = [n]$ extends a preorder \preceq on $[n]$ if

$$i \prec j \text{ implies } i \in I_p \setminus I_{p-1} \text{ and } j \in I_q \setminus I_{q-1} \text{ for some } 1 \leq p < q \leq k,$$

where $i \prec j$ means that $i \preceq j$ and it is not $j \preceq i$.

Let $G = F_{I_1} \cap \dots \cap F_{I_m}$ be a face of P_B and $N_G = \{I_1, \dots, I_m\} \subset B \setminus \{[n]\}$ the corresponding nested set. Note that $\dim G = n - 1 - m$. For each $I \in N_G \cup \{[n]\}$ let I^{root} be the set of roots of I given by

$$I^{\text{root}} = I \setminus \cup \{J \in N_G \mid J \subsetneq I\}.$$

Define a preorder \preceq_G on $[n]$ corresponding to the face $G \subset P_B$ by

$$i \preceq_G j \text{ if and only if } i \in I, j \in I^{\text{root}} \text{ for some } I \in N_G \cup \{[n]\}.$$

We describe the map among face lattices $\pi_{P_B} : L(Pe^{n-1}) \rightarrow L(P_B)$ given by Definition 2.2.

Lemma 5.1. *For a face G of a nestohedron P_B and a flag of subsets \mathcal{F} on $[n]$ we have $\pi_{P_B}(\mathcal{F}) = G$ if and only if a flag \mathcal{F} extends the preorder \preceq_G .*

Proof. The proof follows from the coordinate descriptions of corresponding normal cones. The normal cone σ_G at the face G is given by system of inequalities

$$\sigma_G : x_i \leq x_j \text{ if and only if } i \preceq_G j \text{ for } i, j \in [n].$$

The braid cone $\sigma_{\mathcal{F}}$ corresponding to a flag \mathcal{F} is given by relations (4). We see that $\sigma_{\mathcal{F}} \subset \sigma_G$ if and only if \mathcal{F} extends \preceq_G . \square

For a connected building set B on $[n]$ and a flag of subsets $\mathcal{L} : \emptyset = I_0 \subset I_1 \subset \dots \subset I_k = [n]$ denote by B/\mathcal{L} the *quotient building set*

$$B/\mathcal{L} = \bigoplus_{j=1}^k (B|_{I_j}/I_{j-1}).$$

Define the rank of \mathcal{L} according to B as

$$\text{rk}_B(\mathcal{L}) = \sum_{j=1}^k \text{rk}(B|_{I_j}/I_{j-1}).$$

Note that $\text{rk}_B(\mathcal{L}) = n - c(B/\mathcal{L})$, where $c(B/\mathcal{L})$ is the number of components of the quotient building set B/\mathcal{L} .

Proposition 5.2. *For a connected building set B on $[n]$ and a flag \mathcal{L} of subsets of $[n]$ the ranks $\text{rk}_{P_B}(\mathcal{L})$ and $\text{rk}_B(\mathcal{L})$ coincide*

$$\text{rk}_{P_B}(\mathcal{L}) = \text{rk}_B(\mathcal{L}).$$

Proof. Let G be a proper face of P_B of dimension $\dim G = m$ and $\mathcal{L} : \emptyset = I_0 \subset I_1 \subset \dots \subset I_k = [n]$ be a flag of subsets of $[n]$ such that $\pi_{P_B}(\mathcal{L}) = G$. To the face G corresponds the nested set $N_G = \{J_1, \dots, J_{n-m-1}\} \subset B \setminus \{[n]\}$. A simple argumentation shows that

$$c(B|_{I_p}/I_{p-1}) = |\{J \in N_G \cup \{[n]\} \mid J^{\text{root}} \subset I_p \setminus I_{p-1}\}| \text{ for all } 1 \leq p \leq k.$$

Since for any $J \in N_G \cup \{[n]\}$ there is a unique p such that $J^{\text{root}} \subset I_p \setminus I_{p-1}$ we have

$$\sum_{j=1}^k c(B|_{I_j}/I_{j-1}) = n - m.$$

□

The following theorem gives an algebraic characterization of the quasisymmetric enumerator function $F_q(P_B)$.

Theorem 5.3. *For a building set B the quasisymmetric enumerator function $F_q(P_B)$ associated to a nestohedron P_B coincides with the value of a universal morphism*

$$F_q(P_B) = \Psi_q([B]).$$

Proof. The morphism Ψ_q is given by

$$\Psi_q([B]) = \sum_{\alpha \models \text{gr}(B)} (\zeta_q)_\alpha(B) M_\alpha.$$

By Theorem 3.1 the coefficient corresponding to a composition $\alpha = (i_1, \dots, i_k) \models n$ is determined by

$$(\zeta_q)_\alpha(B) = \sum_{\mathcal{L} : \text{type}(\mathcal{L}) = \alpha} \prod_{j=1}^k q^{\text{rk}(B|_{I_j}/I_{j-1})} = \sum_{\mathcal{L} : \text{type}(\mathcal{L}) = \alpha} q^{\text{rk}_B(\mathcal{L})}, \quad (11)$$

where the sum is over all chains $\mathcal{L} : \emptyset \subset I_1 \subset \dots \subset I_k = V$ of the type α and $\text{rk}_B(\mathcal{L})$ is the rank of the flag \mathcal{L} according to B . Thus

$$F_q(P_B) = \sum_{\mathcal{L}} q^{\text{rk}_B(\mathcal{L})} M_{\text{type}(\mathcal{L})}, \quad (12)$$

where the sum is over all chains of the ground set $[n]$. Theorem follows according to the expansion (6) and Proposition 5.2. \square

Example 5.4. The permutohedron $Pe^{n-1} = P_{B_n}$ is realized as the nestohedron corresponding to the family B_n of all subsets of $[n]$. Since $\text{rk}_{B_n}(\mathcal{L}) = n - |\mathcal{L}|$ for any chain \mathcal{L} of subsets of $[n]$ we have by (12) that

$$F_q(Pe^{n-1}) = \sum_{\mathcal{L}} q^{n-|\mathcal{L}|} M_{\text{type}(\mathcal{L})} = \sum_{\alpha \models n} \binom{n}{\alpha} q^{n-k(\alpha)} M_{\alpha}.$$

Consequently by Theorem 4.4 we derive the well known fact

$$f(Pe^{n-1}, q) = \sum_{\alpha \models n} \binom{n}{\alpha} q^{n-k(\alpha)}.$$

Example 5.5. For the building set $B = \{\{1\}, \dots, \{n\}, [n]\}$ on $[n]$ the corresponding nestohedron is the $(n-1)$ -simplex $P_B = \Delta^{n-1}$. Let \mathcal{L} be a chain of the type $\text{type}(\mathcal{L}) = \alpha \models n$. Obviously $\text{rk}_B(\mathcal{L}) = l(\alpha) - 1$, where $l(\alpha)$ denotes the last component of the composition $\alpha \models n$. Therefore by (11) we have $(\zeta_q)_\alpha(B) = \binom{n}{\alpha} q^{l(\alpha)-1}$, and consequently

$$F_q(\Delta^{n-1}) = \sum_{\alpha \models n} \binom{n}{\alpha} q^{l(\alpha)-1} M_{\alpha}.$$

By rearranging summands according to last components of compositions we obtain

$$F_q(B) = \sum_{k=1}^n \binom{n}{k} q^{k-1} \sum_{\alpha \models n-k} \binom{n-k}{\alpha} M_{(\alpha, k)}.$$

Taking into account that $M_{(1)}^n = \sum_{\alpha \models n} \binom{n}{\alpha} M_{\alpha}$, for each n we have

$$F_q(\Delta^{n-1}) = \sum_{k=1}^n \binom{n}{k} q^{k-1} (M_{(1)}^{n-k})_k.$$

Theorem 4.4 gives the expected

$$f(\Delta^{n-1}, q) = \sum_{k=1}^n \binom{n}{k} q^{k-1} = \frac{(1+q)^n - 1}{q}.$$

5.3 Recursive behavior of $F_q(P_B)$

For a composition $\alpha = (a_1, \dots, a_k)$ and a positive integer r let $(\alpha, r) = (a_1, \dots, a_k, r)$. Define a shifting operator $F \mapsto (F)_r$ on $QSym$ as the linear extension of the map given on the monomial basis by $M_\alpha \mapsto M_{(\alpha, r)}$. Specially $(M_\emptyset)_r = M_{(r)} = x_1^r + x_2^r + \dots$.

The next theorem shows that the weighted quasisymmetric enumerator function for nestohedra $F_q(P_B)$ is determined by recurrence relations.

Theorem 5.6. *The quasisymmetric function $F_q(B) = F_q(P_B)$ is determined by the following recurrence relations*

- (1) $F_q(\bullet) = M_{(1)}$ for the singleton $\bullet = \{\{1\}\}$.
- (2) If $B = B_1 \sqcup B_2$ then $F_q(B) = F_q(B_1)F_q(B_2)$.
- (3) If B is connected then $F_q(B) = \sum_{I \subsetneq [n]} q^{n-|I|-1} (F_q(B|_I))_{n-|I|}$.

Proof. The assertions (1) and (2) are direct consequences of definition of the enumerator $F_q(B)$. It remains to prove the assertion (3). Note that for connected B the contraction B/I remains connected for each $I \subset [n]$ and $\text{rk}(B/I) = n - |I| - 1$. If we rearrange the sum in the expansion (12) according to predecessors of the maximal elements in chains we obtain

$$F_q(B) = \sum_{I \subsetneq [n]} q^{n-|I|-1} \sum_{\mathcal{L}_I} q^{\text{rk}_{B|_I}(\mathcal{L}_I)} M_{(\text{type}(\mathcal{L}_I), n-|I|)},$$

where the last sum is over all chains \mathcal{L}_I of I . This leads, by repeated application of equation (12) to the needed identity. \square

The recursive behavior of the enumerator $F_q(P_B)$ together with Theorem 4.4 reproves the recursive behavior of f -polynomials of nestohedra.

Theorem 5.7 ([16], Theorem 7.11). *The f -polynomial $f(B, q)$ of a nestohedron P_B is determined by the following recurrence relations*

- (1) $f(\bullet, q) = 1$ for the singleton $\bullet = \{\{1\}\}$.
- (2) If $B = B_1 \sqcup B_2$ then $f(B, q) = f(B_1, q)f(B_2, q)$.
- (3) If B is connected then $f(B, q) = \sum_{I \subseteq [n]} q^{n-|I|-1} f(B|_I, q)$.

6 Graph-associahedra

A special class of building sets is produced by simple graphs. The graphical building set $B(\Gamma)$ on a graph Γ is the collection of all subsets of vertices such that induced subgraphs are connected. The polytope $P_\Gamma = P_{B(\Gamma)}$ is called a *graph-associahedron*.

In [11] is considered the following Hopf algebra of graphs. Let \mathcal{G} be a vector space over the field \mathbf{k} spanned by isomorphism classes of simple graphs. It is endowed with a Hopf algebra structure by operations

$$[\Gamma_1] \cdot [\Gamma_2] = [\Gamma_1 \sqcup \Gamma_2] \text{ and } \Delta([\Gamma]) = \sum_{I \subset V} [\Gamma|_I] \otimes [\Gamma/I],$$

where $\Gamma|_I$ is the induced subgraph on I and Γ/I is the induced subgraph on $V \setminus I$ with additional edges uv for all pairs of vertices $u, v \notin I$ connected by edge paths through I . The correspondence $\Gamma \mapsto B(\Gamma)$ defines a Hopf monomorphism from \mathcal{G} to \mathcal{B} .

The induced character on \mathcal{G} over the field of rational functions $\mathbf{k}(q)$ is given by $\zeta_q(\Gamma) = q^{\text{rk}(\Gamma)}$, where $\text{rk}(\Gamma) = \text{gr}(\Gamma) - c(\Gamma)$ with $\text{gr}(\Gamma)$ and $c(\Gamma)$ being the numbers of vertices and connected components of the graph Γ . This results in a q -analog $F_q(\Gamma)$ of the quasisymmetric function invariant $F(\Gamma) = F(P_\Gamma)$ of the graph Γ introduced and studied in [11]. By formula (12) the quasisymmetric function $F_q(\Gamma)$ is described by

$$F_q(\Gamma) = \sum_{\mathcal{L}} q^{\text{rk}(\mathcal{L})} M_{\text{type}(\mathcal{L})}, \quad (13)$$

where $\text{rk}_\Gamma(\mathcal{L}) = \sum_{j=1}^k \text{rk}(\Gamma|_{I_j}/I_{j-1})$ for a chain $\mathcal{L} : \emptyset \subset I_1 \subset \dots \subset I_k = V$.

Theorem 5.6 applied on the case of graph-associahedra gives the following recurrence behavior of $F_q(\Gamma)$.

Proposition 6.1. *The quasisymmetric function $F_q(\Gamma)$ associated to a simple graph Γ satisfies*

- (1) $F_q(\bullet) = M_{(1)}$ for the one-vertex graph,
- (2) If $\Gamma = \Gamma_1 \sqcup \Gamma_2$ then $F_q(\Gamma) = F_q(\Gamma_1)F_q(\Gamma_2)$,
- (3) If Γ is connected then $F_q(\Gamma) = \sum_{I \subseteq V} q^{|V|-|I|-1} (F_q(\Gamma|_I))_{|V|-|I|}$.

The Proposition 6.1 includes the recurrence formula for the special case $q = 0$ obtained in [11, Theorem 7.4]

Corollary 6.2. *The quasisymmetric graph invariant $F(\Gamma)$ satisfies*

- (1) $F(\bullet) = M_{(1)}$ for the one-vertex graph,
- (2) If $\Gamma = \Gamma_1 \sqcup \Gamma_2$ then $F(\Gamma) = F(\Gamma_1)F(\Gamma_2)$,
- (3) If Γ is connected then $F(\Gamma) = \sum_{v \in V} (F_{\Gamma \setminus v})_1$, where $\Gamma \setminus v$ is the induced graph on $V \setminus \{v\}$.

6.1 Graphs with the same weighted enumerator $F_q(\Gamma)$

The following nonisomorphic graphs on six vertices with the same quasisymmetric invariant $F(\Gamma)$ were found in [11, Example 7.5], see Figure 1. We obtain from Corollary 6.2

$$F(\Gamma_1) = F(\Gamma_2) = 720M_{(1,1,1,1,1,1)} + 96M_{(2,1,1,1,1)} + 24M_{(1,2,1,1,1)}.$$

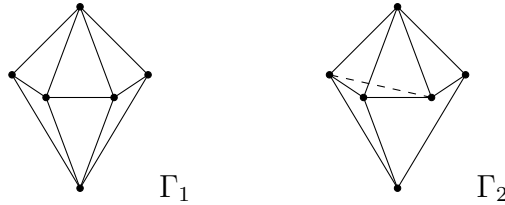


Figure 1: Graphs with $F(\Gamma_1) = F(\Gamma_2)$

By applying Proposition 6.1 we obtain that $F_q(\Gamma)$ of these graphs are also equal. In Table 1 are given coefficients of $F_q(\Gamma_1) = F_q(\Gamma_2)$ by the monomials q^k , $k = 0, 1, 2, 3, 4, 5$.

q^5	$M_{(6)}$
q^4	$6M_{(1,5)} + 6M_{(5,1)} + 11M_{(2,4)} + 15M_{(4,2)} + 18M_{(3,3)}$
q^3	$30M_{(1,1,4)} + 30M_{(1,4,1)} + 30M_{(4,1,1)} + 66M_{(2,2,2)} + 4M_{(2,4)} + 2M_{(3,3)} + 56M_{(1,2,3)} + 60M_{(1,3,2)} + 44M_{(2,1,3)} + 44M_{(2,3,1)} + 54M_{(3,1,2)} + 54M_{(3,2,1)}$
q^2	$108M_{(3,1,1,1)} + 120M_{(1,3,1,1)} + 120M_{(1,1,3,1)} + 120M_{(1,1,1,3)} + 180M_{(1,1,2,2)} + 168M_{(1,2,1,2)} + 168M_{(1,2,2,1)} + 132M_{(2,1,1,2)} + 132M_{(2,1,2,1)} + 132M_{(2,2,1,1)} + 4M_{(1,2,3)} + 16M_{(2,1,3)} + 16M_{(2,3,1)} + 6M_{(3,1,2)} + 6M_{(3,2,1)} + 24M_{(2,2,2)}$
q^1	$360M_{(1,1,1,1,2)} + 360M_{(1,1,1,2,1)} + 360M_{(1,1,2,1,1)} + 336M_{(1,2,1,1,1)} + 264M_{(2,1,1,1,1)} + 12M_{(1,2,1,2)} + 12M_{(1,2,2,1)} + 48M_{(2,1,1,2)} + 48M_{(2,1,2,1)} + 48M_{(2,2,1,1)} + 12M_{(3,1,1,1)}$
q^0	$720M_{(1,1,1,1,1)} + 96M_{(2,1,1,1,1)} + 24M_{(1,2,1,1,1)}$

Table 1: $F_q(\Gamma_1) = F_q(\Gamma_2)$

Theorem 4.4 then implies that f -polynomials of corresponding graph-associahedra are equal. We found

$$f(P_{\Gamma_1}, q) = f(P_{\Gamma_2}, q) = q^5 + 56q^4 + 462q^3 + 1308q^2 + 1500q + 600.$$

Nevertheless the polytopes $P_{\Gamma_i}, i = 1, 2$ are not combinatorially equivalent. We can show this by looking at the 1-skeletons of dual polytopes $G_i = ((P_{\Gamma_i})^*)^{(1)}, i = 1, 2$. These graphs have different sequences of vertex degrees, see Tables 2 and 3. Here d stands for the vertex degree and n for the number of vertices with this degree. Recall that dual polytopes of nestohedra are represented by nested set complexes, see [17, Theorem 6.5]. Therefore $G_i, i = 1, 2$ are graphs whose vertices correspond to connected induced subgraphs of $\Gamma_i, i = 1, 2$ and edges are given by pairs of connected induced subgraphs $\{H_1, H_2\}$ such that either one is contained into another or their union is not connected.

d	30	29	28	26	25	16	15	14	13	12	11
n	4	2	2	2	2	9	9	3	4	6	13

Table 2: Vertex degrees of G_1

d	30	29	28	26	16	15	14	13	12	11
n	2	4	2	4	10	6	4	6	6	12

Table 3: Vertex degrees of G_2

Remark 6.3. By using MathLab programm we found exactly three pairs of six-vertex graphs with the same quasisymmetric function $F(\Gamma)$. If we label adequately the vertices then the sets of edges of graphs given in Figure 1 are $E(\Gamma_1) = \{12, 13, 14, 15, 23, 34, 45, 26, 36, 46, 56\}$ and $E(\Gamma_2) = \{12, 13, 14, 15, 23, 24, 34, 45, 26, 36, 56\}$. Two other pairs are obtained by deleting edges 14 and 14, 26 from Γ_1 and Γ_2 respectively. Similarly as for the pair Γ_1 and Γ_2 it can be shown that the other two pairs have equal q -analogs $F_q(\Gamma)$ and henceforth equal f -polynomials of corresponding P_Γ , which are also combinatorially nonequivalent polytopes.

7 Matroid base polytope

A standard reference monograph for matroid theory is [15]. The most adequate for our purposes is a definition of a matroid as a collection of bases. A matroid $M = ([n], \mathcal{B})$ on the ground set $[n]$ is a nonempty collection \mathcal{B} of subsets of $[n]$ such that the exchange property is satisfied:

- For each $B_1, B_2 \in \mathcal{B}$ and $i \in B_1 \setminus B_2$ there is $j \in B_2 \setminus B_1$ such that $B_1 \setminus \{i\} \cup \{j\} \in \mathcal{B}$.

The members of \mathcal{B} are called bases of M and we denote the collection of bases of M by $\mathcal{B}(M)$. All bases have the same number of elements which is called the rank $r(M)$ of the matroid M .

Definition 7.1. The matroid base polytope is a convex polytope

$$P_M = \text{Conv} \{e_B \mid B \in \mathcal{B}(M)\},$$

where $e_B = \sum_{i \in B} e_i$ and $e_i, i = 1, \dots, n$ are the standard basis vectors in \mathbb{R}^n .

The polytope P_M is contained in the hypersimplex

$$\Delta_r^n = [0, 1]^n \cap \{x_1 + x_2 + \dots + x_n = r(M)\}.$$

The following characterization of matroids in terms of their base polytopes shows that matroid base polytopes are generalized permutohedra.

Theorem 7.2 ([8], Section 2.2, Theorem 1 and [9], Theorem 4.1). *Let \mathcal{S} be a collection of r -subsets of $[n]$ and $P_{\mathcal{S}}$ be a convex polytope in \mathbb{R}^n*

$$P_{\mathcal{S}} = \text{Conv} \{e_S \mid S \in \mathcal{S}\},$$

where $e_S = \sum_{i \in S} e_i$. *Then \mathcal{S} is the collection of bases of a matroid if and only if every edge of $P_{\mathcal{S}}$ is a translate of the vector $e_i - e_j$, for some $i, j \in [n]$.*

Definition 7.3. For a matroid $M = ([n], \mathcal{B})$ the elements $i, j \in [n]$ are equivalent if there exist bases B_1 and B_2 with $i \in B_1$ and $B_2 = (B_1 \setminus \{i\}) \cup \{j\}$. The equivalence classes are called *connected components* of M . The matroid M is *connected* if it has only one connected component.

Denote by $c(M)$ the number of connected components of a matroid M .

Proposition 7.4 ([7], Proposition 2.4). *The dimension of the matroid base polytope P_M corresponding to a matroid M on $[n]$ is determined by*

$$\dim(P_M) = n - c(M).$$

7.1 Flags of matroid base polytope

For a vector $\omega = (\omega_1, \omega_2, \dots, \omega_n) \in \mathbb{Z}_+^n$ we regard the function ω^* on the matroid base polytope P_M as a weight function on a matroid M , so that the ω -weight of a basis $B = \{i_1, \dots, i_{r(M)}\} \in \mathcal{B}(M)$ is given by

$$\omega^*(B) = \langle \omega, e_B \rangle = \omega_{i_1} + \dots + \omega_{i_{r(M)}}.$$

Let $\mathcal{B}_{\omega} \subset \mathcal{B}(M)$ be the collection of bases of M having maximum ω -weight. The collection \mathcal{B}_{ω} consists of all bases obtained by greedy algorithm. The polytope $P_{\mathcal{B}_{\omega}}$ is the face of P_M on which ω^* is maximized. Theorem 7.2 implies that \mathcal{B}_{ω} is the collection of bases of a matroid which we denote by M_{ω} .

Recall that ω determines a unique flag of subsets

$$\mathcal{F}_{\omega} : \emptyset =: F_0 \subset F_1 \subset \dots \subset F_k \subset F_{k+1} := [n] \quad (14)$$

for which ω is constant on $F_i \setminus F_{i-1}$, $i = 1, \dots, k+1$ and satisfies $\omega|_{F_{i+1} \setminus F_i} < \omega|_{F_i \setminus F_{i-1}}$, $i = 1, \dots, k$. The following proposition shows that M_{ω} depends only on the flag \mathcal{F}_{ω} . By $r(A) = r(M|_A)$, where $M|_A$ is the restriction of M to $A \subset [n]$, is defined the rank function of M .

Proposition 7.5 ([3], Proposition 1). *Let $\omega \in \mathbb{Z}_+^n$ and $\mathcal{F} = \mathcal{F}_\omega$ is the corresponding flag given by (14). A base $B \in \mathcal{B}(M)$ of a matroid M has the maximum ω -weight, i.e. $B \in B_\omega$ if and only if $|B \cap (F_i \setminus F_{i-1})| = r(F_i) - r(F_{i-1}), i = 1, \dots, k + 1$.*

We say that a flag \mathcal{F} is maximized at a base $B \in \mathcal{B}(M)$ if and only if $B \in B_\omega$ and $\mathcal{F} = \mathcal{F}_\omega$ for some $\omega \in \mathbb{Z}_+^n$. Denote by M/\mathcal{F} the matroid M_ω corresponding to a flag $\mathcal{F} = \mathcal{F}_\omega$. We determine the map $\pi_{P_M} : L(Pe^{n-1}) \rightarrow L(P_M)$ associated to the matroid base polytope by definition 2.2, namely

$$\pi_{P_M}(\mathcal{F}) = P_{M/\mathcal{F}}.$$

The following proposition describes the matroid M/\mathcal{F} as a sum of its minors.

Proposition 7.6 ([3], Proposition 2). *If $\mathcal{F} = \{\emptyset =: F_0 \subset \dots \subset F_{k+1} := [n]\}$, then*

$$M/\mathcal{F} = \bigoplus_{i=1}^{k+1} (M|_{F_i})/F_{i-1}.$$

Since for the direct sum of matroids we have $P_{M_1 \oplus M_2} = P_{M_1} \times P_{M_2}$, by Proposition 7.4 we obtain

Corollary 7.7. *The P_M -rank of a flag \mathcal{F} is given by*

$$\text{rk}_{P_M}(\mathcal{F}) = \dim(P_{M/\mathcal{F}}) = n - \sum_{i=1}^k c((M|_{F_i})/F_{i-1}).$$

7.2 Quasisymmetric enumerator $F_q(P_M)$ for matroids

The Hopf algebra of matroids $\mathcal{M}at$ was introduced by Schmitt [18] and more intensively studied by Crapo and Schmitt [5], [6].

As a vector space $\mathcal{M}at$ is linearly spanned over the field \mathbf{k} by isomorphism classes of finite matroids $[M]$. The direct sum induces a product, while a coproduct is determined by restrictions and contractions

$$[M_1] \cdot [M_2] = [M_1 \oplus M_2] \quad \text{and} \quad \Delta[M] = \sum_{A \subseteq E} [M|_A] \otimes [M/A].$$

The Hopf algebra $\mathcal{M}at$ is graded $\mathcal{M}at = \bigoplus_{n \geq 0} \mathcal{M}at_n$, where $\mathcal{M}at_n$ denotes the subspace spanned by elements $[M]$ for which the ground set has cardinality n . The Hopf algebra $\mathcal{M}at$ is graded, connected, commutative, but

non-cocommutative. The unit is the class of a unique matroid on the empty set $[M_\emptyset]$.

The character $c : \mathcal{Mat} \rightarrow \mathbf{k}$ defined by

$$c([M]) = \begin{cases} 1, & M \text{ is a direct sum of loops and isthmuses,} \\ 0, & \text{otherwise} \end{cases}$$

turns \mathcal{Mat} into a combinatorial Hopf algebra considered by Billera, Jia and Reiner in [4]. By Theorem 3.1 there is a unique morphism $\Psi : (\mathcal{Mat}, c) \rightarrow (\mathcal{QSym}, \zeta_{\mathcal{Q}})$ of combinatorial Hopf algebras given in monomial bases of quasisymmetric functions by

$$\Psi([M]) = \sum_{\alpha \models n} c_\alpha([M]) M_\alpha.$$

Proposition 7.8 ([4], Proposition 3.3). *The coefficient $c_\alpha([M])$ is the number of flags $\mathcal{F} : \emptyset = F_0 \subset \dots \subset F_{k+1} = [n]$ having $\text{type}(\mathcal{F}) = \alpha$ and for which each subquotient $(M|_{F_i})/F_{i-1}$ is a direct sum of loops and isthmuses.*

We extend the basic field \mathbf{k} into the field of rational functions $\mathbf{k}(q)$ and define the character $c_q : \mathcal{Mat} \rightarrow \mathbf{k}(q)$ with

$$c_q([M]) = q^{\text{rk}(M)},$$

where $\text{rk}(M) = n - c(M)$. Let $\Psi_q : (\mathcal{Mat}, c_q) \rightarrow (\mathcal{QSym}, \zeta_{\mathcal{Q}})$ be a unique morphism of combinatorial Hopf algebras over $\mathbf{k}(q)$.

Theorem 7.9. *For a matroid M the quasisymmetric enumerator function $F_q(P_M)$ associated to the matroid base polytope P_M coincides with the value of a universal morphism*

$$F_q(P_B) = \Psi_q([M]).$$

Proof. The morphism Ψ_q is given by

$$\Psi_q([M]) = \sum_{\alpha \models n} c_{q,\alpha}([M]) M_\alpha,$$

where

$$c_{q,\alpha}([M]) = \sum_{\mathcal{F} : \text{type}(\mathcal{F}) = \alpha} \prod_{i=1}^k q^{\text{rk}((M|_{F_i})/F_{i-1})}.$$

By Corollary 7.7 we have

$$c_{q,\alpha}([M]) = \sum_{\mathcal{F}:\text{type}(\mathcal{F})=\alpha} q^{\text{rk}_{P_M}(\mathcal{F})}.$$

Consequently,

$$\Psi_q([M]) = \sum_{\mathcal{F}} q^{\text{rk}_{P_M}(\mathcal{F})} M_{\text{type}(\mathcal{F})},$$

where the sum is over all flags of $[n]$. This is exactly the form of $F_q(P_M)$ given by identity (6). \square

As a consequence of Theorem 7.9 and Theorem 4.4 we obtain an algebraic description of the f -polynomials of matroid base polytopes.

Theorem 7.10. *The f -polynomial of a matroid base polytope P_M is given by*

$$f(P_M, q) = (-1)^n \mathbf{ps}(\Psi_{-q}([M]))(-1).$$

7.3 The uniform matroid base polytope

We use theorem 7.10 to calculate f -polynomials of uniform matroid base polytopes $P_{U_{n,r}}$. The uniform matroid $U_{n,r}$ is a matroid with the set of bases $\mathcal{B}(U_{n,r}) = \binom{[n]}{r}$ consisting of all r -elements subsets of $[n]$. From definition it follows that $P_{U_{n,r}}$ is the hypersimplex

$$P_{U_{n,r}} = \Delta_r^n.$$

Note that $\mathcal{B}(U_{n,0}) = \{\emptyset\}$ and $\mathcal{B}(U_{n,n}) = \{[n]\}$, so $P_{U_{n,0}} = \mathbf{0}$ and $P_{U_{n,n}} = \mathbf{1}$ are single points, where $\mathbf{0} = (0, \dots, 0)$ and $\mathbf{1} = (1, \dots, 1)$.

Consider the uniform matroid $U_{n,r}$ and assume that $0 < r < n$. Let $\mathcal{F} : \emptyset = F_0 \subset F_1 \subset \dots \subset F_{k+1} = [n]$ be a flag with $\text{type}(\mathcal{F}) = (\alpha_1, \dots, \alpha_i, \alpha_{i+1}, \dots, \alpha_k)$. Define $i_0 = i_0(\mathcal{F}) = \min\{i \mid \alpha_1 + \dots + \alpha_i \geq r\}$ and partial sums $r' = \alpha_1 + \dots + \alpha_{i_0-1}$, $r'' = r' + \alpha_{i_0}$. By proposition 7.6 we find

$$M/\mathcal{F} = U_{r',r'} \oplus U_{\alpha_{i_0},r-r'} \oplus U_{n-r'',0}.$$

It shows that faces of the uniform matroid base polytopes are uniform matroid base polytopes as well, the fact that is obvious from the description

by hypersimplices. We have $P_{M/\mathcal{F}} \cong P_{U_{\alpha_{i_0}, r-r'}}$, which implies the following formula for the P_M -rank, where $M = U_{n,r}$

$$\mathrm{rk}_{P_M}(\mathcal{F}) = \dim(P_{M/\mathcal{F}}) = \begin{cases} 0, & r = r'', \\ \alpha_{i_0} - 1, & r < r'' \end{cases}. \quad (15)$$

Let \circ be the concatenation product defined on monomial bases by

$$M_\alpha \circ M_\beta = M_{\alpha\beta}$$

and linearly extended to the algebra of quasisymmetric functions. We obtain from (15)

$$\begin{aligned} F_q(P_{U_{n,r}}) &= \binom{n}{r} (M_1)^r \circ (M_1)^{n-r} + \\ &+ \sum_{0 \leq r' < r < r' + \lambda \leq n} \binom{n}{r'} \binom{n-r'}{\lambda} q^{\lambda-1} (M_1)^{r'} \circ M_\lambda \circ (M_1)^{n-r'-\lambda}. \end{aligned}$$

The principal specialization evaluated at -1 gives

$$f(P_{U_{n,r}}, q) = \binom{n}{r} + \sum_{0 \leq r' < r < r' + \lambda \leq n} \binom{n}{r'} \binom{n-r'}{\lambda} q^{\lambda-1}.$$

It determines the f -vector of $P_{U_{n,r}}$ in terms of trinomial coefficients

$$f_0 = \binom{n}{r} \quad \text{and} \quad f_{k-1} = \sum_{0 \leq r' < r < r' + k \leq n} \binom{n}{r', k, n-r'-k} \quad \text{for } k = 2, \dots, n.$$

Example 7.11. The polytope $P_{U_{4,2}} = \Delta_2^4$ is an octahedron. We obtain

$$\begin{aligned} F_q(P_{U_{4,2}}) &= 6 (M_{(2,2)} + 2M_{(1,1,2)} + 2M_{(2,1,1)} + 4M_{(1,1,1,1)}) q^0 + 12M_{(1,2,1)} q + \\ &+ 4 (M_{(1,3)} + M_{(3,1)}) q^2 + M_{(4)} q^3, \end{aligned}$$

which yields to the expected expression $f(P_{U_{4,2}}, q) = 6 + 12q + 8q^2 + q^3$.

8 Properties of $F_q(M)$

In the following we will write $F_q(M)$ instead of $F_q(P_M)$. We examine some algebraic properties of the quasisymmetric enumerator $F_q(M)$. First we give an example which shows that $F_q(M)$ contains more information about matroids than its specialization at $q = 0$. The following example of matroids with the same quasisymmetric invariant $F(M)$ is borrowed from [4, Example 8.1].

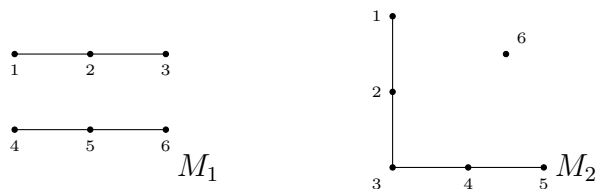


Figure 2: Matroids with $F(M_1) = F(M_2)$

Example 8.1. Let M_1 and M_2 be affine matroids depicted on Figure 2. M_1 and M_2 are the rank 3 matroids on the set $[6]$ having every triple but $\{1, 2, 3\}$, $\{4, 5, 6\}$ and $\{1, 2, 3\}$, $\{3, 4, 5\}$ as bases, respectively. Consider flags $\emptyset \subset \{i\} \subset [6] \setminus \{j\} \subset [6]$, $1 \leq i \neq j \leq 6$ of the type $(1, 4, 1)$. All minors $M_1|_{[6] \setminus \{j\} / \{i\}}$ are connected, so we have the summand $30q^3 M_{(1,4,1)}$ in $F_q(M_1)$. On the other hand the flag $\emptyset \subset \{3\} \subset \{1, 2, 3, 4, 5\} \subset [6]$ produces a two components minor of M_2 which contributes with $q^2 M_{(1,4,1)}$ in $F_q(M_2)$. It shows that $F_q(M_1) \neq F_q(M_2)$.

8.1 Matroid duality

For a matroid M on $[n]$ with the collection of bases $\mathcal{B}(M)$ the dual matroid M^* is defined by the set of bases $\mathcal{B}(M^*) = \{[n] \setminus B \mid B \in \mathcal{B}(M)\}$. The affine transformation $\text{aff} : [0, 1]^n \rightarrow [0, 1]^n$ of the n -cube $\text{aff}(x) = \mathbf{1} - x$, $x \in [0, 1]^n$, where $\mathbf{1} = (1, \dots, 1)$ determines an affine isomorphism between P_M and P_{M^*} . The behavior of the quasisymmetric invariant $F_q(M)$ for $q = 0$ under matroid duality is determined by [4, Proposition 4.1]. We generalize this to the q -analog $F_q(M)$.

Lemma 8.2. *For a matroid M and a flag \mathcal{F} on its ground set $[n]$ it holds*

$$\text{aff}(P_{M/\mathcal{F}}) = P_{M^*/\mathcal{F}^{op}}.$$

Proof. The statement follows from the fact that the flag \mathcal{F} is maximized at a base $B \in \mathcal{B}(M)$ if and only if \mathcal{F}^{op} is maximized at the cobase $[n] \setminus B \in \mathcal{B}(M^*)$. \square

Proposition 8.3. *The quasisymmetric function $F_q(M)$ satisfies $F_q(M) = \sum_{\alpha \models n} p_\alpha(q) M_\alpha$ if and only if $F_q(M^*) = \sum_{\alpha \models n} p_\alpha(q) M_{\text{rev}(\alpha)}$.*

Proof. We use the expansion (7)

$$F_q(M^*) = \sum_{G \in L(P_{M^*})} q^{\dim(G)} \sum_{\mathcal{F}: P_{M^*(\mathcal{F})} = G} M_{\text{type}(\mathcal{F})}.$$

Denote by $G^* = \text{aff}(G)$. Then by lemma 8.2 and the fact $\dim(G) = \dim(G^*)$ we can write

$$F_q(M^*) = \sum_{G^* \in L(P_M)} q^{\dim(G^*)} \sum_{\mathcal{F}: P_{M(\mathcal{F}^{op})} = G^*} M_{\text{type}(\mathcal{F}^{op})}.$$

The proposition follows from (8). \square

8.2 Valuation

For matroids $M_1 = ([n], \mathcal{B}_1)$ and $M_2 = ([n], \mathcal{B}_2)$ of the same rank r we can define $M_1 \cap M_2 = ([n], \mathcal{B}_1 \cap \mathcal{B}_2)$ which is usually not a matroid, but if polytopes P_{M_1} and P_{M_2} meet along a common nonempty face, the intersection $M_1 \cap M_2$ will be a matroid of rank r and $P_{M_1} \cap P_{M_2} = P_{M_1 \cap M_2}$, see [4, Proposition 7.2]. A matroid base polytope decomposition is a decomposition $P_M = \sqcup_{i=1}^k P_{M_i}$ into matroid base polytopes P_{M_i} , $i = 1, \dots, k$ such that $P_{M_i} \cap P_{M_j}$ is a common face of both P_{M_i} and P_{M_j} for each $i \neq j$. Specially, for $k = 2$, a decomposition is called a hyperplane split.

The enumerator $F_q(Q)$ for generalized permutohedra lacks to be valiative, contrary to its specialization $q = 0$, see [4, Theorem 9.2]. We illustrate this by the following example based on [4, Remark 8.5].

Example 8.4. Let M_1 be an affine matroid of rank 3 depicted on Figure 2 and M_2, M_3 be matroids given by the sets of basis $\mathcal{B}(M_2) = \{123, 124, 125, 126, 134, 135, 136, 234, 235, 236\}$, $\mathcal{B}(M_3) =$

{145, 146, 156, 245, 246, 256, 345, 346, 356, 456}. Then $M_2 \cong M_3$, $M_1 \cap M_2 \cong M_1 \cap M_3 \cong U_{2,3} \oplus U_{1,3}$ and we have a decomposition of the hypersimplex $P_{U_{3,6}} = P_{M_1} \sqcup P_{M_2} \sqcup P_{M_3}$. We can check that

$$F_q(U_{3,6}) \neq F_q(M_1) + F_q(M_2) + F_q(M_3) - F_q(M_1 \cap M_2) - F_q(M_1 \cap M_3).$$

9 Conclusion

The weighted quasisymmetric enumerator $F_q(Q)$ associated to generalized permutohedra has two main meanings. Geometrically it is the enumerator of lattice points of the normal fan of a generalized permutohedron and this aspect was originated by Billera, Jia and Reiner in [4]. Algebraically it appears as a result of two universal objects, the first one is the Hopf monoid of generalized permutohedra GP and the second one is the combinatorial Hopf algebra of quasisymmetric functions $QSym$. This categorical and algebraic approach is originated by Aguiar, Bergeron and Sottile in [2] and more recently by Aguiar and Ardila in [1]. The ubiquity of generalized permutohedra comes from their equivalence with submodular functions. Properly defined submodular functions associated to combinatorial objects give their embeddings as Hopf submonoids to the Hopf monoid GP. Composing with the functor that takes Hopf monoids to Hopf algebras and with universal morphism to quasisymmetric functions produce the invariant $F_q(Q)$. Aguiar and Ardila obtained a simple formula for antipode of GP which can help in further study of properties of the enumerator $F_q(Q)$. Apart from that we can exploit the idea of universality of the Hopf monoid GP to obtain enumerative invariants of other combinatorial objects such as posets, hypergraphs, simplicial complexes etc. Also it would be interesting to try to extend this lattice points enumerator to other types of generalized Coxeter permutohedra.

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