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CLOSED FORM SOLUTIONS FOR THE STABILITY AND FREE VIBRATION ANALYSIS OF LAMINATED COMPOSITE PLATES

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ABSTRACT: Analytical formulations and solutions to fundamental frequency and buckling loads analysis of simply supported isotropic and layered anisotropic cross-ply composite plates are presented. The displacement model based on Generalized Laminate Plate Theory (**GLPT**) assumes piece-wise linear variation of in-plane displacement components and constant transverse displacement through thickness of the plate. It also includes the quadratic variation of transverse shear stresses within each layer of the plate. The large deflection theory (in Von Karman sense) is incorporated into the buckling analysis. The equations of motion are obtained using Hamilton's principle. Closed form solution is derived following the Navier's technique and by solving the eigenvalue problem. The effects of side-to-thickness ratio, aspect ratio, coupling between bending and stretching and number of layers on the fundamental frequencies and critical buckling loads are investigated. The results of the presented theory (**GLPT**) are compared with the exact **3D** elasticity theory, Higher-order Shear Deformation Theory (**HSDT**) and Classical Laminated Plate Theory (**CLPT**) solutions. The study concludes that the present model accurately predicts fundamental frequencies and buckling loads of composite plates, while the **CLPT** is inadequate for the analysis of non-homogeneous laminated plates.

Key words: composite plates, analytical solutions, fundamental frequencies, buckling loads, assessment

1. Introduction

Due to their high strength-to-weight ratio, high stiffness-to-weight ration, added to their excellent fatigue strength, ease of formability, high damping, resistance to corrosion and wide range of operating temperatures, structural components made of laminated composite materials have a wide variety of applications such as in aircraft and aerospace industry, automobile industry, sporting goods, offshore structures, and civil engineering applications. To use them efficiently a good understanding of their structural and dynamical behavior under various loading conditions are needed. Namely, during their build and fabrication, composites are subjected to various in-plane loads which can lead to rapid failure. Also, an understanding of the composites behavior under dynamic loading is essential because the loading can cause severe damage of the composites, such as matrix cracking, fiber cracking and delamination.

A laminate is a multilayered composite made up of several individual layers (laminae), in each of which the fibers are oriented in a predetermined direction to provide efficiently the required strength and stiffness parameters. The greater differences in elastic properties between fiber and matrix materials lead to a high ratio of in-plane Young's modulus to transverse shear modulus, resulting that the transverse shear deformations are much pronounced for laminated plates than for isotropic plates. Thus, the **C**lassical **L**aminate **P**late **T**heory (**CLPT**), which ignores the effect of transverse shear deformation, becomes inadequate for the analysis of multilayered composites. Namely, the buckling loads and free vibration frequencies of **CLPT** are higher than those obtained from shear deformation theories. Therefore a variety of refined shear deformable plate theories are proposed to model both thin and thick plates in a single formulation. The **F**irst order **S**hear **D**eformation **T**heory (**FSDT**), based on Mindlin Reissner assumptions, yields a constant shear strain variation through the thickness and thus requires the use of shear correction factors, in order to approximate the quadratic distribution in the elasticity theory. It is also known that the accurate prediction of shear correction factors for anisotropic laminates is cumbersome and problem dependent. A compromising less expensive model can be achieved by using single layer models based on **H**igher order **S**hear **D**eformation **T**heories (**HSDT**). These models involve higher order expansions of displacement field in powers of the thickness coordinate, giving the quadratic variation of out-of plane strains and therefore not requiring the use of artificial shear correction factors. Pioneering work on the structural analysis formulation for closed form or discrete solutions based on **HSDT** can be reviewed in Phan [1] & Reddy [1], Mallikarjuna & Kant, Owen [2] & Li [2] among others. All the above mentioned theories are also called the equivalent-single-layer laminate theories (**ESL** theories), because they use a single function to describe the displacement field across the plate thickness. Even they give acceptable results for both thin and thick plate behavior, with less computational costs, compared to 3D elasticity theory, their application is limited to global response of laminated components such as gross deflections, critical buckling loads, and fundamental frequencies. However, when a local response at the ply level is needed, such as near materials and geometrical discontinuities, the analysis require the use of **3D** or Layerwise Plate Theories (**LWT**), which use full 3D kinematics and constitutive relations. Srinavas, Rao and Noor presented exact three dimensional solutions for the free vibration of isotropic, orthotropic and anisotropic composite laminated plates, which serve as benchmark solutions for many researchers. Despite its accuracy, the large number of variables makes **3D** theory less practical, when analyzing large regions, thus giving the priority to **LWT** theories.

The objective of this paper is to present the **LWT** of Reddy [1] also called the **G**eneralized **L**ayerwise **P**late **T**heory (**GLPT**), applied to buckling and free vibration analysis of laminated composite plates. Analytical solutions are derived for simply supported cross-ply laminated plate and compared with **3D** elasticity solutions, **CLPT** and **HSDT** solutions. The effects of side-to-thickness ratio, aspect ratio, coupling between bending and stretching and number of layers on the fundamental frequencies and critical buckling loads are investigated. The study has shown that the proposed model gives excellent results when compared to **HSDT** and **3D** elasticity theory and that **CLPT** becomes inadequate for the analysis of plates made of advanced filamentary composite materials.

2. Theoretical formulation

2.1 Displacement field

Consider a laminated plate (Fig. 1) composed of n orthotropic laminae. The integer k denotes the layer number that starts from the plate bottom. Plate middle surface coordinates are (x, y, z), while layer coordinates are (x_k, y_k, z_k). Plate and layer thickness are denoted as h and h_k, respectively. We assume that:

- the layers are perfectly bonded together,
- the material of each layer is linearly elastic and has three planes of materials symmetry (i.e., orthotropic),
- strains are small,
- each layer is of uniform thickness,
- the inextensibility of normal is imposed.

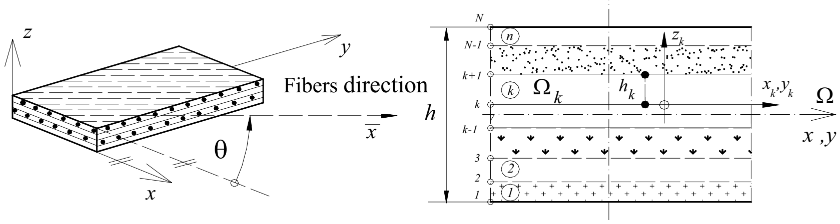

Fig. 1. Multilayer laminated plate

The displacements components (u_1, u_2, u_3) at a point (x, y, z) can be written as:

$$
\begin{aligned}
u_1(x,y,z) &= u(x,y) + \sum_{I=1}^{N} U^I(x,y)\cdot\Phi^I(z) \\
u_2(x,y,z) &= v(x,y) + \sum_{I=1}^{N} V^I(x,y)\cdot\Phi^I(z)\,, \\
u_3(x,y,z) &= w(x,y)
\end{aligned}
\tag{1}
$$

where (u, v, w) are the displacements of a point $(x, y, 0)$ on the reference plane of the laminate, U^I and V^I are undetermined coefficients, and $\Phi^I(z)$ are layerwise continuous functions of the thickness coordinate. In the view of finite element approximation, the functions $\Phi^I(z)$ are the one-dimensional (linear, quadratic or cubic) Lagrange interpolation functions of the thickness coordinates and (U^I, V^I) are the values of (u_1, u_2) at the I-th plane. If we assume linear Lagrange interpolation of in-plane displacement components through the thickness, it could be recognized that each layer is in fact a 1D finite element and that the in-plane displacements are piece-wise continuous through the laminate thickness.

2.2 Strain displacement relations

The strains associated with the displacement field (1) can be computed using the von Karman strain-displacement relation to include the geometric nonlinearities, as follows:

$$
\begin{aligned}
\varepsilon_{xx} &= \frac{\partial u_1}{\partial x} + \underline{\frac{1}{2}\left(\frac{\partial u_3}{\partial x}\right)^2} = \frac{\partial u}{\partial x} + \sum_{I=1}^{N}\frac{\partial U^I}{\partial x}\Phi^I + \underline{\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2}\,, \\
\varepsilon_{yy} &= \frac{\partial u_2}{\partial y} + \underline{\frac{1}{2}\left(\frac{\partial u_3}{\partial y}\right)^2} = \frac{\partial v}{\partial y} + \sum_{I=1}^{N}\frac{\partial V^I}{\partial y}\Phi^I + \underline{\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2}\,, \\
\gamma_{xy} &= \frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} + \underline{\frac{\partial u_3}{\partial x}\frac{\partial u_3}{\partial y}} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \sum_{I=1}^{N}\left(\frac{\partial U^I}{\partial y} + \frac{\partial V^I}{\partial x}\right)\Phi^I + \underline{\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}}\,,
\end{aligned}
\tag{2}
$$

$$\gamma_{xz} = \frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x} = \sum_{I=1}^{N} U^I \frac{d\Phi^I}{dz} + \frac{\partial w}{\partial x}, \quad \gamma_{yz} = \frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial y} = \sum_{I=1}^{N} V^I \frac{d\Phi^I}{dz} + \frac{\partial w}{\partial y}.$$

2.3 Constitutive equations

The stress-strain relations in the laminate coordinates can be written as:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix}^{(k)} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & 0 & 0 \\ Q_{12} & Q_{22} & Q_{23} & 0 & 0 \\ Q_{13} & Q_{23} & Q_{33} & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & Q_{45} \\ 0 & 0 & 0 & Q_{45} & Q_{55} \end{bmatrix}^{(k)} \times \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}^{(k)} . \quad (3)$$

where $\boldsymbol{\sigma}^{(k)} = \{\sigma_{xx} \quad \sigma_{yy} \quad \tau_{xy} \quad \tau_{xz} \quad \tau_{yz}\}^{(k)T}$ and $\boldsymbol{\varepsilon}^{(k)} = \{\varepsilon_{xx} \quad \varepsilon_{yy} \quad \gamma_{xy} \quad \gamma_{xz} \quad \gamma_{yz}\}^{(k)T}$ are stress and strain components, respectively, of k-th laminae in global coordinates.

2.5 Virtual work statement

The virtual work statement can be written using Hamilton's principle, by neglecting the body forces as:

$$\int_0^t \int_\Omega \Bigg\{ N_{xx}\left(\frac{\partial \delta u}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial x}\right) + N_{yy}\left(\frac{\partial \delta v}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial \delta w}{\partial y}\right) + N_{xy}\left(\frac{\partial \delta u}{\partial y} + \frac{\partial \delta v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial \delta w}{\partial x}\right) +$$
$$+ Q_x \frac{\partial \delta w}{\partial x} + Q_y \frac{\partial \delta w}{\partial y} + \sum_{I=1}^{N}\left[N_{xx}^I \frac{\partial \delta U^I}{\partial x} + N_{yy}^I \frac{\partial \delta V^I}{\partial y} + N_{xy}^I\left(\frac{\partial \delta U^I}{\partial y} + \frac{\partial \delta V^I}{\partial x}\right) + Q_x^I U^I + Q_y^I V^I\right] -$$
$$- I_0(\dot{u}\delta\dot{u} + \dot{v}\delta\dot{v} + \dot{w}\delta\dot{w}) - \sum_{I=1}^{N} I^I\left(\dot{U}^I\delta\dot{u} + \dot{u}\delta\dot{U}^I + \dot{V}^I\delta\dot{v} + \dot{v}\delta\dot{V}^I\right) - \sum_{I=1}^{N}\sum_{J=1}^{N} I^{IJ}\left(\dot{U}^I\delta\dot{U}^J + \dot{V}^I\delta\dot{V}^J\right)\Bigg\} d\Omega \; dt = 0 \quad (4)$$

From (4) we can easily obtain the Euler-Lagrange equations of motion of **GLPT** by integrating by parts the derivatives of various quantities and collecting the coefficients of $\delta u, \delta v, \delta w, \delta U^I, \delta V^I$ separately:

$$\begin{aligned}
\delta u = 0: &\quad N_{xx,x} + N_{xy,y} = I_0\ddot{u} + \sum_{J=1}^{N} I^J \ddot{U}^J \\
\delta v = 0: &\quad N_{xy,x} + N_{yy,y} = I_0\ddot{v} + \sum_{J=1}^{N} I^J \ddot{V}^J \\
\delta w = 0: &\quad Q_{x,x} + Q_{y,y} + \eta(w) = I_0\ddot{w} \\
\delta U^I = 0: &\quad N^I{}_{xx,x} + N^I{}_{xy,y} - Q^I{}_x = I^I\ddot{u} + \sum_{J=1}^{N} I^{IJ}\ddot{U}^J \\
\delta V^I = 0: &\quad N^I{}_{xy,x} + N^I{}_{yy,y} - Q^I{}_y = I^I\ddot{v} + \sum_{J=1}^{N} I^{IJ}\ddot{V}^J
\end{aligned} \qquad I = 1, \dots N \quad (5)$$

where $I_0 = \int_{-h/2}^{h/2} \rho \, dz, \quad I^I = \int_{-h/2}^{h/2} \rho \, \Phi^I dz, \quad I^{IJ} = \int_{-h/2}^{h/2} \rho \, \Phi^I \Phi^J dz$, ρ is mass density, and

$$\eta(w) = \frac{\partial}{\partial x}\left(N_{xx}\frac{\partial w}{\partial x} + N_{xy}\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial y}\left(N_y\frac{\partial w}{\partial y} + N_{xy}\frac{\partial w}{\partial x}\right).$$

3. The Navier's solutions

3.1 Free vibration analysis

The analytical solution for the free vibration of laminated composite plates using **GLPT** can be developed for the case of rectangular cross-ply plates axb with simply supported boundary conditions:

$$v = w = V^1 = N_{xx} = N_{xx}^1 = 0 \quad \text{at} \quad x = 0, a.$$

$$u = w = U^1 = N_{yy} = N_{yy}^1 = 0 \quad \text{at} \quad y = 0, b. \tag{6}$$

The following displacements satisfy the boundary conditions and the Euler-Lagrange equations of motion:

$$\begin{aligned}
&\left(u(x,y); U^1(x,y)\right) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(X_{mn}; R^1{}_{mn}\right)\cdot e^{i\omega_{mn}t}\cdot\cos\frac{m\pi}{a}x\cdot\sin\frac{n\pi}{b}y\,,\\
&\left(v(x,y); V^1(x,y)\right) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(Y_{mn}; S^1{}_{mn}\right)\cdot e^{i\omega_{mn}t}\cdot\sin\frac{m\pi}{a}x\cdot\cos\frac{n\pi}{b}y\,,\\
&w(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{mn}\cdot e^{i\omega_{mn}t}\cdot\sin\frac{m\pi}{a}x\cdot\sin\frac{n\pi}{b}y\,,
\end{aligned} \tag{7}$$

Following the standard Navier procedure, and using only the liner terms in strain-displacement relations (2), the governing differential equations (5) are transformed into an algebraic system for each mode (m,n):

$$\left([\mathbf{K}] - \omega_{mn}^2\cdot[\mathbf{M}]\right)\{\mathbf{\Delta}\} = \{\mathbf{0}\}. \tag{8}$$

where ω_{mn} are vibration frequencies, and $\{\mathbf{\Delta}\}^T = \left\{X_{mn}, Y_{mn}, W_{mn}, R_{mn}^1, S_{mn}^1\right\}$ is vector of mode shapes. The lowest value of ω_{11} is the *fundamental frequency*.

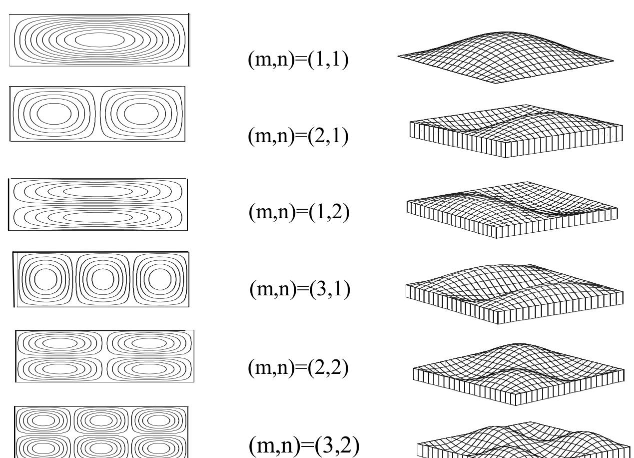

Fig. 2. The contour lines and mode shapes for simply supported rectangular plate

3.2 Buckling analysis

For the buckling problem $e^{i\omega_{mn}t}$ is neglected in (7) and full strain-displacement relations (2) are used. Substituting assumed displacement field into the governing differential equations (5), and assuming that the plate is subjected to only in-plane compressive forces $N_{xx} = -\lambda\hat{N}_{xx}$, $\hat{N}_{yy} = -\lambda\hat{N}_{yy}$, $\hat{N}_{xy} = -\lambda\hat{N}_{xy}$,using the Navier's technique we obtain the following set of algebraic equations:

$$([\mathbf{K}] - \lambda_{mn} \cdot [\mathbf{G}])\{\mathbf{\Delta}\} = \{\mathbf{0}\} \tag{9}$$

Thus, for each choice of m and n we obtain the *characteristic numbers* or *eigenvalues* λ_{mn} and the corresponding nonzero solutions, called *characteristic functions* or *eigenfunctions*. The smallest of all not equal to zero is the *critical value* λ_{cr}. Then the *critical load* is $N_{cr} = -\lambda_{cr}\hat{N}$.

4. Numerical results and discussion

Numerical results are presented for the stability and free vibration analysis of rectangular, simply supported, cross-ply laminated plates. The individual layers are taken to be of equal thickness and each layer of a plate is a unidirectional reinforced composite, made of one of the following materials:

Material I: $E_1 / E_2 = 25, G_{12} / E_2 = 0.5, G_{13} / E_2 = 0.5, G_{23} / E_2 = 0.2, \nu_{12} = \nu_{13} = \nu_{23} = 0.25$,

Material II: $E_1 / E_2 = 40, G_{12} / E_2 = 0.6, G_{13} / E_2 = 0.5, G_{23} / E_2 = 0.5, \nu_{12} = \nu_{13} = \nu_{23} = 0.25$.

It is assumed that both materials have $\rho = 1$ and $E_2 = 1$. The vibration frequencies and buckling loads are given in non-dimensionalized form $\overline{w} = w a^2 / h (\rho / E_2)^{1/2}$, $\overline{N} = N_{cr} b^2 / (E_2 h^3)$.

4.1 Free vibration analysis

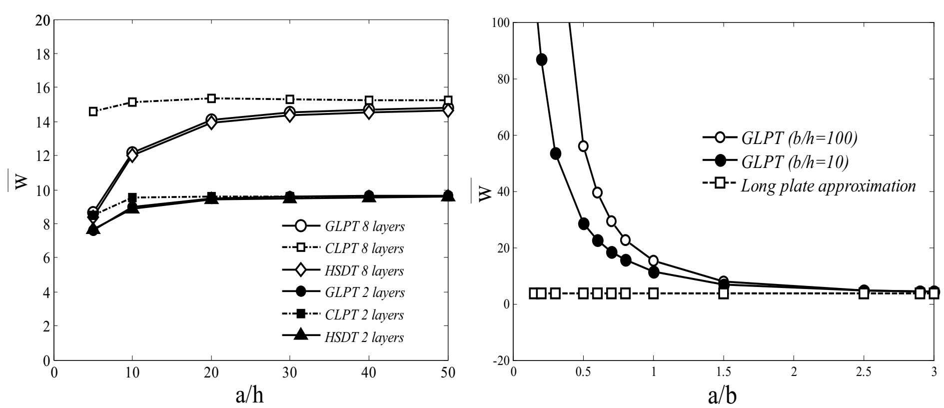

Fig. 3. Material I: a) The effect of number of layers and thickness on the fundamental frequencies of cross-ply (0/90/0…) laminated plates b) The effect of aspect ratio on the fundamental frequency of vibration for thin and thick cross-ply (0/90/0) laminated plate.

From the Figure 3a it is obvious that the fundamental frequencies significantly increase as a/h increases in thick plate range, but the increase is small beyond $a/h = 20$. It can be also seen that the differences between **GLPT** and **HSDT** are negligible, and that the differences between

GLPT, **HSDT** and **CLPT** becomes smaller as we reach the thin plate limit. For very thin plates fundamental frequencies are practically constant. The Figure 3b shows the variation of fundamental frequencies with the aspect ratio, while keeping the value of 'b' constant. We conclude that the fundamental frequency becomes smaller, as we approach to the long plate approximation, for both thin and thick plates. This is due to the decrease in stiffness of the plate with increasing the aspect ratio.

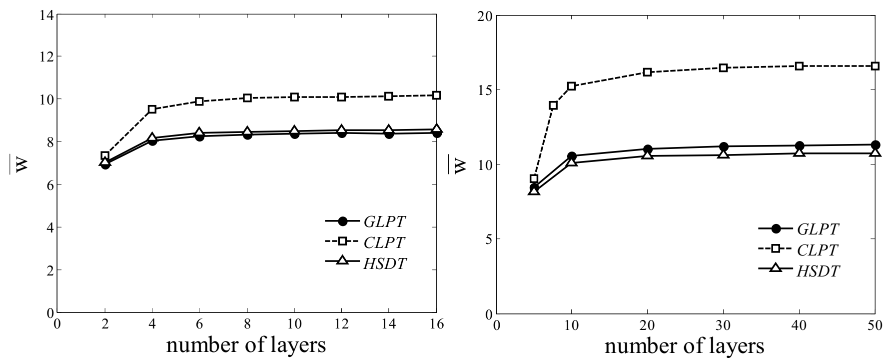

Fig. 4. Material II: The effect of coupling between bending-stretching on the fundamental frequency of cross-ply (0/90/0), $a/h = 5$ laminated plate a) $E_1/E_2 = 10$ b) $E_1/E_2 = 40$

As the modulus ratio have significant effect on the shearing deformation, the differences between **CLPT** and **GLPT** becomes greater for the plate with $E_1/E_2 = 40$ on Figure b, compared to $E_1/E_2 = 10$ on Figure a.

Source	No of layers	E_1/E_2				
		3	10	20	30	40
3D	3	0.2647	0.3284	0.3824	0.4109	0.4301
Present		0.2621	0.3261	0.3785	0.4036	0.4203
Error^{3D}		1.0%	0.7%	1.0%	1.8%	2.3%
CLPT		0.2920	0.4126	0.5404	0.6434	0.7320
3D	5	0.2659	0.3409	0.3979	0.4314	0.4537
Present		0.2698	0.3406	0.3950	0.4272	0.4371
Error^{3D}		1.5%	0.1%	0.7%	1.0%	1.1%
CLPT		0.2920	0.4126	0.5404	0.6434	0.7320
3D	4	0.2618	0.3258	0.3762	0.4066	0.4301
Present		0.2666	0.3292	0.3807	0.4124	0.4334
Error^{3D}		1.8%	1.1%	1.2%	1.4%	0.8%
CLPT		0.2868	0.3888	0.4991	0.5890	0.6669
3D	6	0.2644	0.3366	0.3936	0.4278	0.4509
Present		0.2683	0.3381	0.3948	0.4294	0.4529
Error^{3D}		1.5%	0.4%	0.3%	0.4%	0.4%
CLPT		0.2897	0.4022	0.5223	0.6196	0.7036

Table 1. Effect of orthotropy of individual layers on the fundamental frequency of simply supported square composite plates with $a/h=5$, $\overline{\overline{w}} = \overline{w}(h/a)^2$ symmetric and anti-symmetric laminates

The effects of orthotropy and coupling between bending and stretching on the fundamental frequencies are shown in Table 1 (Material II). As expected, the decrease in bending stretching coupling, with an increase in number of layers, increases the fundamental frequencies. Also the bending stretching coupling effects for anti-symmetric laminates can not be ignored even for low modulus ratio. The **CLPT** overestimates the fundamental frequencies, especially with the increase in the degree of orthotropy. The results predicted by **GLPT** are in good agreement with **3D** elasticity solution.

4.2 Buckling analysis

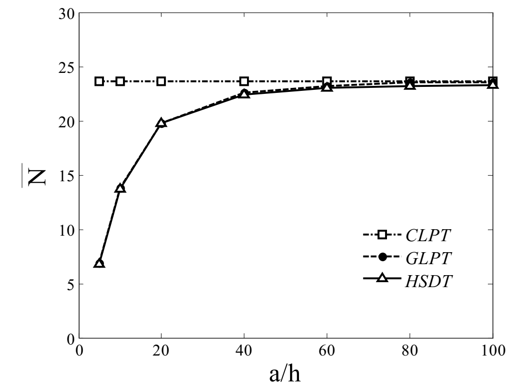

Fig. 5. Material I: The thickness effect on the critical buckling load of cross-ply, square (0/90/0), laminated plate under uniaxial compression

The Figure 5 shows that the **CLPT** theory overestimates the buckling load. For very thick plates ($a/h < 5$) the **CLPT** error is much as 350%. For thin plates, however, the **CLPT** predicts the buckling loads reasonably accurate.

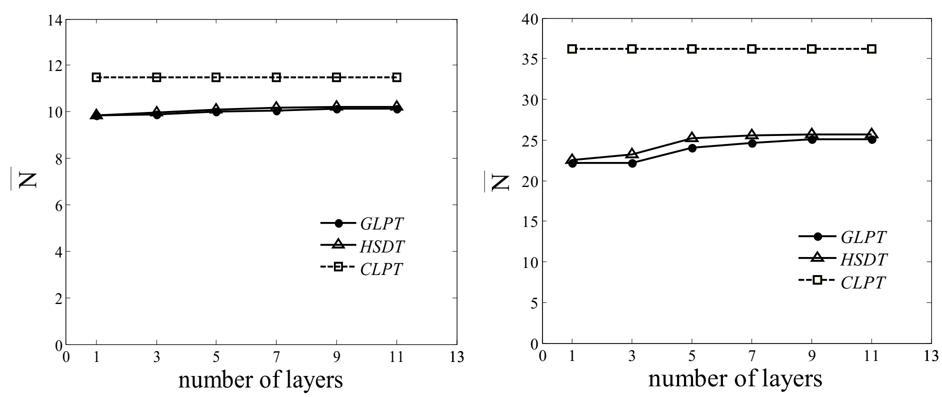

Fig. 6. Material II: The effect of coupling between bending-stretching on the buckling loads of cross-ply, square (0/90/0), $a/h = 5$ laminated plate under uniaxial compression a) $E_1/E_2 = 10$ b) $E_1/E_2 = 40$

The effects of bending stretching coupling on the buckling loads is reduced with the increasing the number of layers (Figure 6). With the increase in modulus ratio, the effect of shearing deformation becomes dominant, thus giving the advantage to share deformation theories compared to classical plate theories (**CLPT**).

Source	No of layers	E_1/E_2				
		3	10	20	30	40
Anti-symmetric laminates						
Noor	2	4.6948	6.1181	7.8196	9.3746	10.8167
Present		4.7910	6.2524	7.9769	9.5476	11.0019
$Error^{3D}$		2.0%	2.2%	2.0%	1.8%	1.7%
CPT		4.6948	6.1181	7.8196	9.3746	10.8167
Noor	4	5.1738	9.0164	13.7429	17.7829	21.2796
Present		5.2349	9.1222	13.8984	17.9785	21.5076
$Error^{3D}$		1.2%	1.2%	1.1%	1.1%	1.7%
CPT		5.5738	10.2950	16.9880	23.6750	30.3590
Noor	6	5.2673	9.6051	15.0014	19.6394	23.6689
Present		5.3169	9.6643	15.0313	19.6103	23.5643
$Error^{3D}$		0.9%	0.6%	0.2%	0.1%	0.4%
CPT		5.6738	10.9600	18.5020	26.0420	35.5820
Symmetric laminates						
Noor	3	5.3044	9.7621	15.0191	19.3040	22.8807
Present		5.3901	9.8413	14.9122	18.9085	22.1513
$Error^{3D}$		1.6%	0.8%	0.7%	2.1%	3.2%
CPT		5.7538	11.4920	19.7120	27.9360	36.1600
Noor	5	5.3255	9.9603	15.6527	20.4663	24.5929
Present		5.3853	9.9986	15.5379	20.1305	24.0112
$Error^{3D}$		1.1%	0.4%	0.7%	1.7%	2.4%
CPT		5.7538	11.4920	19.7120	27.936	36.1600

Table 2. Effect of orthotropy of individual layers on the critical buckling loads of simply supported square composite plates with *a/h=10*, $\overline{N} = N_{cr} b^2 / \left(E_2 h^3\right)$ anti-symmetric and symmetric laminates

In Table 2, the effects of degree of orthotropy and number of layers on the critical buckling loads are investigated for simply supported bidirectional laminates, under uniaxial compression (Material II) for *a/h=10*. As expected, higher buckling loads are obtained with increasing number of layers and module ratio. Again, the **CLPT** over predicts the buckling loads and **GLPT** results are in very good agreement with **3D** closed form solution **(** Noor).

The Figure 7 (Material I) shows the variation of the buckling load as a function of the aspect ratio for different values of (m,n). It is shown that for short and broad plates (*a/b<1*) a minimum value of the critical force is obtained for m=1. For (*a/b<<1*), that is for very short and broad plates, the ratio *a/b* can be neglected, because the critical force does not depend on the plate width and depends only on its length. Namely, for each value of (m,n) the lowest buckling load is the same as the buckling load of the corresponding long plate approximation.

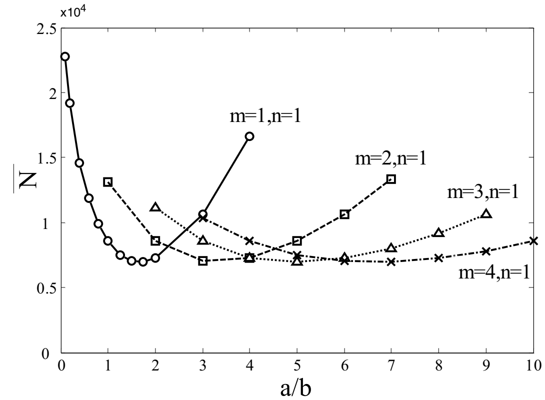

Fig. 7. Material I: The variation of the buckling load as a function of the aspect ratio for m=1, 2, 3, 4 of cross-ply, square (0/90/0), $h/b = 0.1$ laminated plate under uniaxial compression

5. Conclusion

Analytical formulations and solutions for stability and free vibration analysis of laminated composite plates, using the Generalized Laminated Theory is presented. The theory accounts for the layer-wise variation of in-plane displacement field and parabolic distribution of transverse shear stresses within each layer, thus being able to accurately model both global and local response of laminated plates.

The analytical solution for the stability and free vibration analysis are derived for the case of cross-ply, simply supported laminated plates, using the Navier's technique and by solving the eigenvalue problem. It has been shown that the effects of side-to-thickness ratio, aspect ratio, coupling between bending and stretching and number of layers, play important role on the fundamental frequencies and critical buckling loads. The study has also concluded that the proposed **GLPT** model gives excellent results compared to **3D** elasticity solutions, while the **C**lassical **L**aminated **P**late **T**heory (**CLPT)** is inadequate even for thin non-homogeneous laminated plates. Namely, thin plate assumption increases stiffness of the plate and therefore yields to lower deflections, higher frequencies and higher buckling loads. In general, the degree of non-homogeneity has a significant effect on the stability and free vibration response of laminated composite plates.

The obtained results will serve as a benchmark for the future development of discrete and nonlinear models of laminated plates, based on Generalized Laminated Plate Theory.

References

[6] Reddy, J.N.,Phan, N.D., *Stability and vibration of isotropic, orthotropic and laminated plates according to a higher-order shear deformation theory*, Journal of Sound and Vibration, Vol. 98(2), 1985., pp. 157-170.

[7] Owen, D.R.J.,Li, Z.H., *A refined analysis of Laminared Plates by Finite Element Displacement Methods-II. Vibration and Stability*, Computers and Structures, Vol. 26(6), 1987., pp. 915-923.

[8] Vuksanovic, Dj., Cetkovic, M., *Analytical Solution For Multilayer Plates Using General Layerwise Plate Theory,* Facta Universitatis, Vol. 3(2), 2005., pp. 121-136.

[9] Cetkovic, M., *Primena Metode Konacnih Elemenata na Opstu Teoriju Laminatnih Ploca*, Magistarska teza, Gradjevinski Fakultet u Beogradu, 2005.