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APPLICATION OF INTEGRAL TRANSFORM METHOD TO CALCULATE IMPEDANCE FUNCTIONS

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Abstract. To solve vibration problems of structure founded on the soil, the dynamic behavior of the soil needs to be understood and an accurate dynamic stiffness model of the soil has to be developed. Frequency dependent dynamic stiffness matrix of the massless, flexible soil-structure interface can be calculated analytically or numerically, depending on the complexity of the problem, using Boundary Element Method [1] or Thin Layer Method [3]. In this paper the impedance functions of a stiff rectangular foundation laying on a half-space are determined with the help of the Integral Transform Method (ITM) [4]. The Integral Transform Method is an efficient method to calculate wave propagation in an elastic homogeneous, or layered half-space. By the use of the decomposition of Helmholtz, the Lamé's equations of elastodynamics are converted to a system of decoupled partial differential wave equations in space-time domain. With the help of a threefold Fourier Transform in the wave number-frequency domain wave equations can be transformed into a system of three decoupled ordinary differential equations which can be solved in the transformed domain. The results in the original domain can be finally obtained by an Inverse Fourier Transform. Using ITM method the dynamic stiffness of flexible foundation are determinate first. After that the impedance functions of the stiff foundation are obtained using kinematic transformation matrix. The obtained results are compared with impedance functions from literature.

1. Introduction

1.1. Impedance

Impedance can be any kind of resistance to wave oscillation. For example, electrical impedance can be calculated as a ratio between voltage and current, acoustic impedance as a ratio between sound pressure and particle velocity, etc. For the purpose of this paper, mechanical impedance is calculated as a ratio between force and response quantity, where the response quantity is displacement

$$Z_M = \frac{force}{response\ quantity} \tag{1}$$

1.2. History of Impedance Functions

Impedance functions are frequency dependent foundation dynamic stiffnesses functions used in the dynamic soil-structure interaction problems. Those functions were first introduced by *Lamb*, 1904. He studied the vibrations of a linear elastic half-space due to a harmonic load acting on a point. In 1936, *Reissner* analyzed the response to a vertical harmonic excitation of a plate placed at the surface of a homogeneous elastic half-space. He was the first to notice the existence of energy dissipated by radiation. Between 1953 and 1956, *Sung*, *Quilan*, *Arnold* and *Bycroft* were working on generalization of the work of *Reissner* by introducing the six degrees of freedom of the footing. Ten years later, *Hsieh* and *Lysmer* introduced for the first time the idea that soil - footing vibrations in vertical direction can be represented with a single-degree-of-freedom system which stiffness and damping are independent of frequency - *Lysmer's analogy*. This approach was extended to all degrees of freedom by *Richart* and *Whitman*. In order to solve soil-structure interaction problems, many numerical approaches are being developed form the beginning of 60's [6]. The most successful ones are *Finite Element Method - FEM* and *Boundary Element Method - BEM*. The impedance functions for different type of foundations could be obtained using one of the before mention methods [5], [7].

1.3. Integral Transform Method

It is clear that FEM is widely applicable and efficient, but there are some fields where is not very suitable to use FEM. For example, while analyzing the behavior of a layered half-space due to a dynamic loading, as the soil is semi-infinite and some kind of boundary conditions are needed to account Sommerfeld's radiation conditions, it is more convenient to use *Integral Transform Method* (ITM). ITM is based on solving the Lamé's elastodynamics equations of half-space using the Helmholtz potentials and Fourier transformations. It is very efficient solution technique which leads to a better understanding of the physical nature of the problem, which can be integrated into FEM or BEM approaches [2]. On the other hands ITM has a very restricted domain for application.

2. Propagation of Waves in Continuum

In general, the system of equations of motion of an elastic continuum is nonlinear, but, many wave propagation effects in elastic solids can be adequately described by a linearized theory. The system of equations governing the motion of linearly elastic a homogeneous isotropic solid are obtained from the stress-equation of motion, Hooke's law and strain-displacement relations, in the form

$$\mu u_{i,jj} + (\lambda + \mu) u_{j,ji} = \rho \ddot{u}_i \tag{2}$$

where λ and μ are two material parameters known as *Lamé*'s constants

$$\mu = \frac{E}{2(1+\nu)}; \qquad \lambda = \nu \frac{E}{(1+\nu)(1-2\nu)} \tag{3}$$

while E is *Young*'s modulus and ν is *Poisson*'s ratio.
In vector notation the equation (2) can be written as:

$$(\lambda + \mu)\nabla\nabla \cdot \boldsymbol{u} + \mu \nabla^2 \boldsymbol{u} = \rho \ddot{\boldsymbol{u}} \tag{4}$$

The equilibrium equations, the kinematic and constitutive relations and hence, the *Navier*'s equation, must be satisfied at every interior point of the undeformed body, i.e. in the domain Ω. On the surface S of the undeformed body, boundary conditions must be satisfied, also.
The system of equations (4) couples the three displacement components. It can be uncoupled using *Helmholtz* decomposition, which states that any vector $\boldsymbol{u}$ can be written as a sum of gradient of a scalar potential $\phi(x,t)$ and the curl of a vector potential $\boldsymbol{\Psi}(x,t)$ as:

$$\boldsymbol{u} = \nabla\phi + \nabla\times\boldsymbol{\Psi} \tag{5}$$

The scalar potential $\phi(x,t)$ and the components of vector potential $\Psi_i(x,t),\ i = x,y,z$ are coupled through the boundary conditions.
Substitution of (5) into the field equation (4) yields

$$\mu\nabla^2[\nabla\phi + \nabla\times\boldsymbol{\Psi}] + (\lambda+\mu)\nabla\nabla\cdot[\nabla\phi + \nabla\times\boldsymbol{\Psi}] = \rho[\nabla\ddot{\phi} + \nabla\times\ddot{\boldsymbol{\Psi}}] \tag{6}$$

Since that $\nabla\cdot\nabla\times\boldsymbol{\Psi} = 0$, one obtains upon rearranging terms

$$\nabla[(\lambda+2\mu)\nabla^2\phi] + \nabla\times\mu\nabla^2\boldsymbol{\Psi} = \rho\nabla\ddot{\phi} + \rho\nabla\times\ddot{\boldsymbol{\Psi}} \tag{7}$$

The displacement representation (5) satisfies the equation of motion if

$$\nabla^2\phi - \frac{1}{c_p^2}\ddot{\phi} = 0 \tag{8}$$

and

$$\nabla^2\boldsymbol{\Psi} - \frac{1}{c_s^2}\ddot{\boldsymbol{\Psi}} = 0 \tag{9}$$

In these equations of motion c_p is the velocity of *dilatational (longitudinal) wave* or *P-wave*:

$$c_p = \sqrt{\frac{\lambda+2\mu}{\rho}} \tag{10}$$

c_s is the velocity of *distorsional (shear) wave* or *S-wave*:

$$c_s = \sqrt{\frac{\mu}{\rho}} \tag{11}$$

and ∇^2 is the Laplacian:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{12}$$

The equation (5) can be written in the matrix form

$$\begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}\phi + \begin{bmatrix} 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix}\begin{bmatrix} \Psi_x \\ \Psi_y \\ \Psi_z \end{bmatrix} \tag{13}$$

The four potential fields ϕ, Ψ_x, Ψ_y and Ψ_z are not uniquely determined by the three displacements u_x, u_y and u_z. Usually, but not always, the relation $\nabla\cdot\boldsymbol{\Psi} = 0$, is taken as the additional constraint condition. Here, as a special case, Ψ_z is set to zero. Then, the equation (13) can be written as

$$\begin{aligned} u_x &= \phi_{,x} - \Psi_{y,z} \\ u_y &= \phi_{,y} + \Psi_{x,y} \\ u_z &= \phi_{,z} - \Psi_{x,y} + \Psi_{y,x} \end{aligned} \tag{14}$$

3. Solution of Wave Equations using Integral Transform Method

To solve the equations of motions (8) and (9) the *Integral Transform Method* (ITM) together with the *Fourier Transform* will be used. The procedure is schematically described in figure 1.

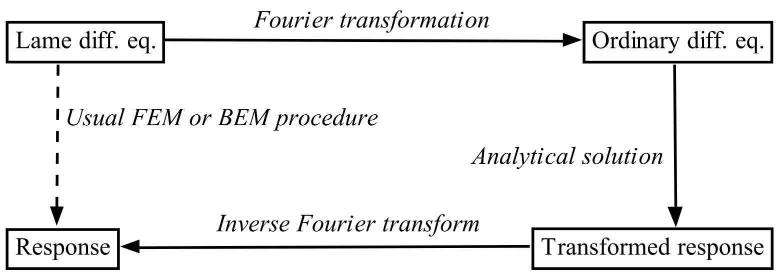


Figure 1. ITM procedure scheme

The *Fourier Transform* $\hat{f}(\omega)$ of a function $f(t)$ and *Inverse Fourier Transform* are defined by the integrals:

$$\hat{f}(\omega) = \int_{-\infty}^{+\infty} f(t)e^{-i\omega t}dt \quad \circ - \bullet \quad f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{f}(\omega)e^{i\omega t}d\omega \tag{15}$$

where $\circ - \bullet$ sign represents the *Fourier Transformation*.
In case of a function with several independent variables, multiple integrals are used, concerning the transformation of each variable

$$\begin{gathered}\hat{f}(k_x,k_y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x)e^{-i(k_x x+k_y y)}dxdy \\ \circ \\ | \\ \bullet \\ f(x,y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \hat{f}(k_x,k_y)e^{i(k_x x+k_y y)}dk_x dk_y\end{gathered} \tag{16}$$

By a threefold *Fourier Transform* with regard to $x \circ - \bullet k_x$, $y \circ - \bullet k_y$ and $t \circ - \bullet \omega$, equations (8) and (9) can be transformed into the ordinary differential equations regarding the z-direction in wave number domain

$$\left(\frac{\omega^2}{c_p^2} - k_x^2 - k_y^2\right)\hat{\phi} + \frac{\partial^2\hat{\phi}}{\partial z^2} = 0 \tag{17}$$

$$\left(\frac{\omega^2}{c_s^2} - k_x^2 - k_y^2\right)\hat{\Psi}_i + \frac{\partial^2\hat{\Psi}_i}{\partial z^2} = 0 \quad , \quad i = x,y \tag{18}$$

The solution of the equations (17) and (18) is

$$\hat{\phi} = A_1 e^{\lambda_1 z} + A_2 e^{-\lambda_1 z} \tag{19}$$

$$\hat{\Psi}_i = B_{1i}e^{\lambda_2 z} + B_{2i}e^{-\lambda_2 z} \quad , \quad i = x,y \tag{20}$$

where A_1, A_2, B_{1i} and B_{2i} are the constants of integration, which can be obtained from boundary conditions, while λ_1 and λ_2 are equal to

$$\lambda_1^2 = k_x^2 + k_y^2 - k_p^2, \qquad \lambda_2^2 = k_x^2 + k_y^2 - k_s^2 \tag{21}$$

In equations (21) k_p and k_s are wave numbers for *P*- and *S-waves*

$$k_p = \frac{\omega}{c_p}, \qquad k_s = \frac{\omega}{c_s} \tag{22}$$

This solution allows to derivative macro-element relations for each layer between the stress and displacement at the top and bottom boundary of the layer in transformed domain. In transformed domain the equation (14) can be written as

$$\begin{array}{l} \hat{u_x} = ik_x\hat{\phi} - \hat{\Psi}_{y,z} \\ \hat{u_y} = ik_y\hat{\phi} - \hat{\Psi}_{x,z} \\ \hat{u_y} = \hat{\phi}_{,z} - ik_y\hat{\Psi}_x + ik_x\hat{\Psi}_y \end{array} \tag{23}$$

Substituting equations (20) and (19) into equation (23) the displacement vector in transformed domain is obtained in the form

$$\begin{bmatrix} \hat{u_x} \\ \hat{u_y} \\ \hat{u_z} \end{bmatrix} = \begin{bmatrix} ik_x & ik_y & 0 & 0 & -\lambda_2 & \lambda_2 \\ ik_y & ik_x & \lambda_2 & -\lambda_2 & 0 & 0 \\ \lambda_1 & -\lambda_1 & -ik_y & -ik_y & ik_x & ik_x \end{bmatrix} \cdot \{C\} \tag{24}$$

where

$$\{C\}^T = \begin{bmatrix} A_1e^{z\lambda_1} & A_2e^{-z\lambda_1} & B_{1x}e^{z\lambda_2} & B_{2x}e^{-z\lambda_2} & B_{1y}e^{z\lambda_2} & B_{2y}e^{-z\lambda_2} \end{bmatrix} \tag{25}$$

The stress vector in transformed domain can be obtained from the strain-displacement relations and *Hooke*'s law [9] as

$$\begin{bmatrix} \hat{\sigma}_x \\ \hat{\sigma}_y \\ \hat{\sigma}_z \\ \hat{\tau}_{xy} \\ \hat{\tau}_{yz} \\ \hat{\tau}_{zx} \end{bmatrix} = \mu \begin{bmatrix} -2kx^2 - \frac{\lambda_1}{\mu}k_p^2 & -2kx^2 - \frac{\lambda_1}{\mu}k_p^2 & 0 & 0 & -2ik_x\lambda_2 & 2ik_x\lambda_2 \\ -2ky^2 - \frac{\lambda_1}{\mu}k_p^2 & -2ky^2 - \frac{\lambda_1}{\mu}k_p^2 & 2ik_y\lambda_2 & -2ik_x\lambda_2 & 0 & 0 \\ 2k_r - k_s^2 & 2k_r - k_s^2 & -2ik_y\lambda_2 & 2ik_y\lambda_2 & 2ik_x\lambda_2 & -2ik_x\lambda_2 \\ -2k_xk_y & -2k_xk_y & ik_x\lambda_2 & -ik_x\lambda_2 & -ik_y\lambda_2 & ik_y\lambda_2 \\ 2ik_y\lambda_1 & -2ik_y\lambda_1 & \lambda_2^2 + k_y^2 & \lambda_2^2 + k_y^2 & -k_xk_y & -k_xk_y \\ 2ik_x\lambda_1 & -2ik_x\lambda_1 & k_xk_y & k_xk_y & -(\lambda_2^2 + k_x^2) & -(\lambda_2^2 + k_x^2) \end{bmatrix} \tag{26}$$

In the case of a layered half-space, it is better to use a new constants $\overline{A_1}$, $\overline{B_{1i}}$ instead of A_1, B_{1i} according to

$$\begin{array}{l} A_1e^{\lambda_1 z} = A_1e^{\lambda_1 h}e^{-\lambda_1 h}e^{\lambda_1 z} = \overline{A_1}e^{\lambda_1(z-h)} \\ B_{1i}e^{\lambda_2 z} = B_{1i}e^{\lambda_2 h}e^{-\lambda_2 h}e^{\lambda_2 z} = \overline{B_{1i}}e^{\lambda_2(z-h)} \end{array} \tag{27}$$

where h is depth of the layer and $h > z$. The displacement vector, for each layer, in the transformed domain can be now written as

$$\begin{bmatrix} \hat{u_x} \\ \hat{u_y} \\ \hat{u_z} \end{bmatrix} = \begin{bmatrix} ik_x & ik_y & 0 & 0 & -\lambda_2 & \lambda_2 \\ ik_y & ik_x & \lambda_2 & -\lambda_2 & 0 & 0 \\ \lambda_1 & -\lambda_1 & -ik_y & -ik_y & ik_x & ik_x \end{bmatrix} \cdot \{\overline{C}\} \tag{28}$$

were $\{\overline{C}\}^T$ is

$$\{\overline{C}\}^T = \begin{bmatrix} A_1e^{(z-h)\lambda_1} & A_2e^{-z\lambda_1} & B_{1x}e^{(z-h)\lambda_2} & B_{2x}e^{-z\lambda_2} & B_{1y}e^{(z-h)\lambda_2} & B_{2y}e^{-z\lambda_2} \end{bmatrix} \tag{29}$$

The unknown integration constants can be obtained from the boundary conditions at the interface between the layers. At the upper surface of the top element the boundary conditions of the half space must be fulfilled, as well as the Sommerfeld's radiation condition if the bottom element goes to infinity.

4. Half-Space Displacements Due to Harmonic Unit Force

In order to get the impedance functions of the rectangular foundations lying on the half-space, the displacements of the half-space due to unit harmonic forces acting in vertical direction z, and both horizontal directions, x and y (Figure 2) have to be calculated.

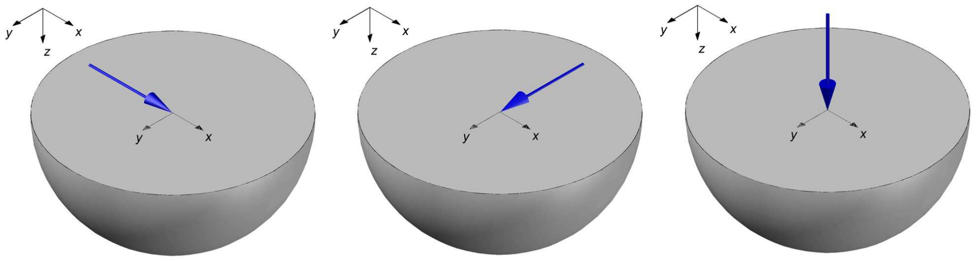


Figure 2. Harmonic force acting on the surface of the half-space

On the figures below, the displacements at the surface of the half-space due to the vertical and horizontal force of unit amplitude, at frequency $\omega = 50\ Hz$ are displayed.

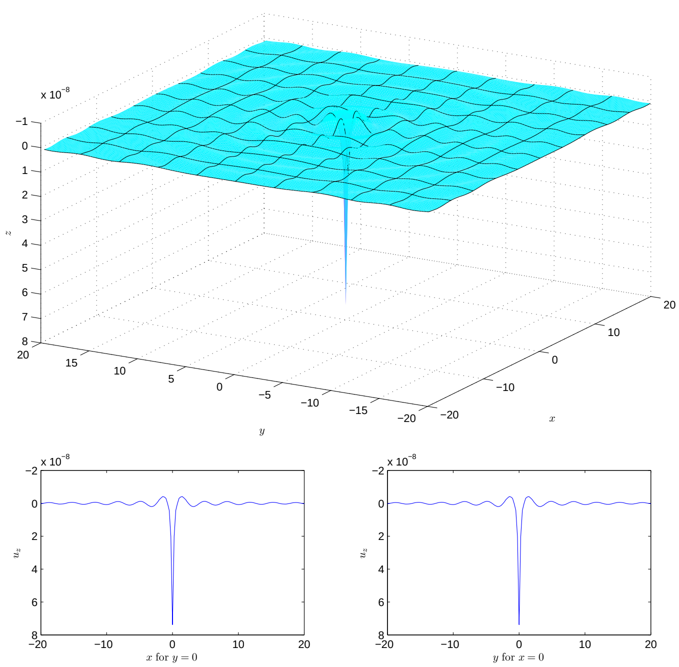


Figure 3. Vertical displacements $u_z\ (m)$, $P_z = 1\ (kN)$, $\omega = 160\ rad/s$, $\nu = 0.4$

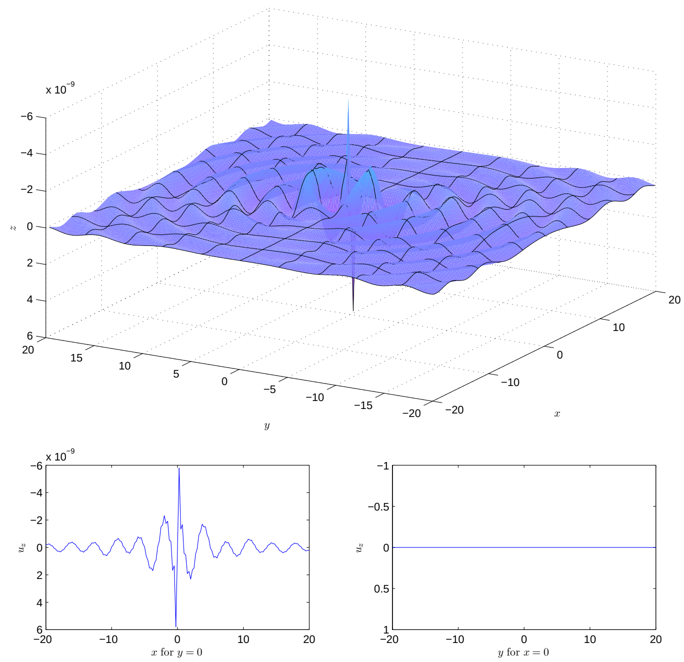


Figure 4. Horizontal displacements u_x (m), $P_z = 1$ (kN), $\omega = 160$ rad/s, $\nu = 0.4$

5. Impedance Functions

5.1. Dynamic stiffness matrix of flexible rectangular foundation

The dynamic stiffness matrix of the flexible rectangular foundation, $\mathbf{K}_f$, is obtained by inverting the dynamic flexibility matrix, $\mathbf{K}_f = \mathbf{F}_f^{-1}$. Elements of dynamic flexibility matrix $\mathbf{F}_f$ represent the nodal displacements at the surface of the half-space due to corresponding harmonic forces of a unit amplitude. They are obtained using ITM. If $n \times n$ is a number of nodes of a rectangular surface on half-space, the dimension of the flexibility matrix is $3n \times 3n$. Nodal displacements vector $u_f(3n, 1)$ and corresponding force vector $P_f(3n, 1)$ are related by dynamic stiffness matrix of flexible foundation $\mathbf{K}_f(3n, 3n)$

$$\mathbf{P}_f = \mathbf{K}_f \mathbf{u}_f \tag{30}$$

5.2. Dynamic stiffness matrix of rigid rectangular foundation

Dynamic stiffness matrix of the corresponding rigid, massless, rectangular foundation is obtained from dynamic stiffness matrix of flexible foundation using kinematic transformation. Rigid foundation has 6 degrees of freedom: three translator vibrations, in x, y and z directions, and three rotational vibrations, around x, y and z axes. The vector of

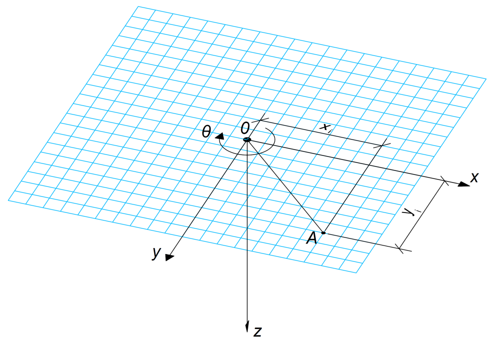


Figure 5. Interaction surface between rigid foundation and soil

displacements $\mathbf{u}_r$ in the centroid O of the interaction surface and corresponding force vector $\mathbf{P}_r$ are, respectively

$$\mathbf{u}_r = \begin{bmatrix} u_x \\ u_y \\ u_z \\ \varphi_x \\ \varphi_y \\ \varphi_z \end{bmatrix} \qquad \mathbf{P}_r = \begin{bmatrix} P_x \\ P_y \\ P_z \\ M_x \\ M_y \\ M_z \end{bmatrix} \tag{31}$$

Vectors $\mathbf{u}_r$ and $\mathbf{P}_r$ are related by

$$\mathbf{P}_r = \mathbf{K}_r \mathbf{u}_r \tag{32}$$

where $\mathbf{K}_r(6,6)$ is dynamic stiffness matrix of rigid foundation.
Vector of nodal displacements $\mathbf{u}_f$ and vector $\mathbf{u}_r$ are relate with kinematic constraint equation

$$\mathbf{u}_f = \mathbf{a}\mathbf{u}_r \tag{33}$$

where $\mathbf{a}(3(n \times n), 6)$ is kinematic matrix

$$\mathbf{a} = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_i \\ \vdots \\ \mathbf{a}_{n \times n} \end{bmatrix} \tag{34}$$

This matrix consists of $n \times n$ sub-matrices $\mathbf{a}_i$, $i = 1, 2, \cdots n \times n$. Each sub-matrix $\mathbf{a}_i$ is obtained from kinematic consideration, regarding node $A = i$ and centroid of foundation O (Figure 5), as

$$\mathbf{a}_i = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -y_i \\ 0 & 1 & 0 & 0 & 0 & x_i \\ 0 & 0 & 1 & y_i & -x_i & 0 \end{bmatrix} \tag{35}$$

Quantities x_i and y_i are horizontal distances in x and y direction, respectively, between the centroid of the surface foundation O and node $A = i$, Figure 5.
Equaling the energy of the deformation, expressed in term of both pairs of variables,

$$E = \mathbf{P}_f^T \mathbf{u}_f = \mathbf{P}_r^T \mathbf{u}_r \tag{36}$$

and taking into account Eqs. (30), (32) and (33), obtained is the relation between the dynamic stiffness matrix of rigid and flexible foundation in the form

$$\mathbf{K}_r = \mathbf{a}^T \mathbf{K}_f \mathbf{a} \tag{37}$$

At least, the dynamic stiffness matrix of rigid rectangular foundation is obtained as

$$\mathbf{K}_r = \begin{bmatrix} K_{xx} & 0 & 0 & 0 & K_{x,my} & 0 \\ 0 & K_{yy} & 0 & K_{y,mx} & 0 & 0 \\ 0 & 0 & K_{zz} & 0 & 0 & 0 \\ 0 & K_{mx,y} & 0 & K_{mx} & 0 & 0 \\ K_{my,x} & 0 & 0 & 0 & K_{my} & 0 \\ 0 & 0 & 0 & 0 & 0 & K_{mz} \end{bmatrix} \tag{38}$$

The dynamic stiffness matrix $\mathbf{K}_r$ is frequency dependent, complex matrix, which can be written as a sum of real and imaginary part

$$\mathbf{K}_r(a_0) = R_e(\mathbf{K}_r(a_0)) + i \cdot I_m(\mathbf{K}_r(a_0)) \tag{39}$$

The impedance functions are functions representing the dimensionless real and imaginary part of dynamic stiffness matrix $\mathbf{K}_r$. Real part represents the dynamic stiffness, while imaginary part represents damping of foundation in one direction. These functions are usually written as functions of dimensionless frequency $a_0 = \omega B/c_s$, where B is the foundation half-width. To obtain impedances the dynamic stiffnesses are reduced by the appropriate coefficients of reduction. For vertical and horizontal stiffness the reduction coefficient is equal to GB, while for rocking and torsional stiffness it is equal to GB^3, where G is shear modulus of the soil.

6. Numerical Example

In the following example the impedance functions of the rigid massless square foundation lying on the half-space is calculated using ITM. The dimensions of square foundation are $5m \times 5m$. The foundation is divided into the mesh, with unit 0.25 meters in both directions. The half-space characteristics are

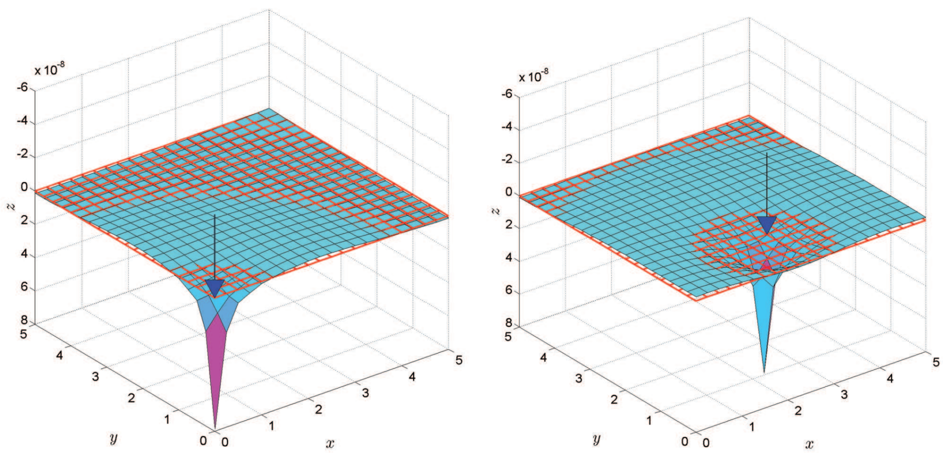
- Elastic modulus: $E = 5 \cdot 10^7 (1 + 2iD) \quad kN/m^2$
- Damping ratio: $D = 0.02$
- Poisson coefficient: $\nu = 0.4$
- Density: $\rho = 2000 \quad kg/m^3$

The process of obtaining impedance functions is divided into several parts.
First, the displacements at the surface of the half-space (in x, y and z directions) for single unit force in x, y and z direction were calculated for every chosen angular frequency ω, using ITM. The discretization mesh should be wide and dense enough to avoid troubles with singularities and aliasing [8]. The obtained displacement fields are shown at Figures 3 and 4.

The next step is calculation of flexibility matrix of the flexible foundation, F_f. The displacement fields is calculated for one node of the mesh and than shifted accordingly to the global coordinate system, in order to fill the flexibility matrix F_f. Figure 6 shows the example of shifting data for filling the columns and rows of the flexibility matrix F_f that correspond to the vertical displacements due to vertical harmonic unit force. Assume that displacement field is calculated for the acting force node which has index (i, j), Figure 6.a. The displacement field due to the force acting in node (m,n), Figure 6.b, can be obtained for every pair of index increments k and l from the relation

$$u_z(i+k, j+l) = u_z(m+k, n+l) \tag{40}$$

The stiffness matrix of the flexible foundation K_f is obtained by inverting the flexibility matrix, $K_f = F_f^{-1}$.



(a) (b)

Figure 6. Calculation of the flexibility matrix. Shifting the displacements field data.

Finally, the stiffness matrix of the rigid foundation K_r is obtained from the stiffness matrix of the flexible foundation K_f using kinematic transformation defined in Eq. (37). Once K_r is calculated for every chosen ω the impedance functions can be obtained, as described at the end of the section 5.

Figures 7, 8, 9, 10 represent impedance functions. Since the foundation is square, $K_{yy} = K_{xx}$ and $K_{my} = K_{mx}$. Dashed line refers to the results obtained in the numerical example described in this section; solid line refers to the results obtained by *Schmid* [5] using *BEM*. Impedance functions obtained using ITM have the same shape but higher values than functions obtained using *BEM*.

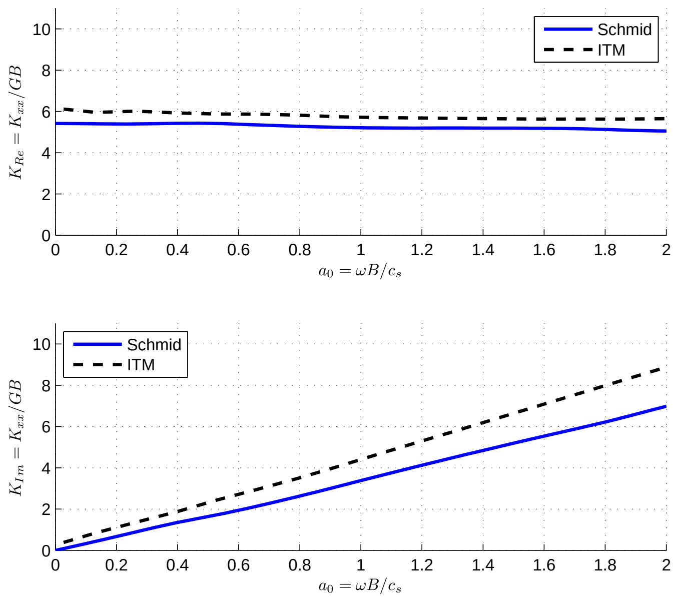


Figure 7. Horizontal dynamic stiffness K_{xx}, $\nu = 0.4$

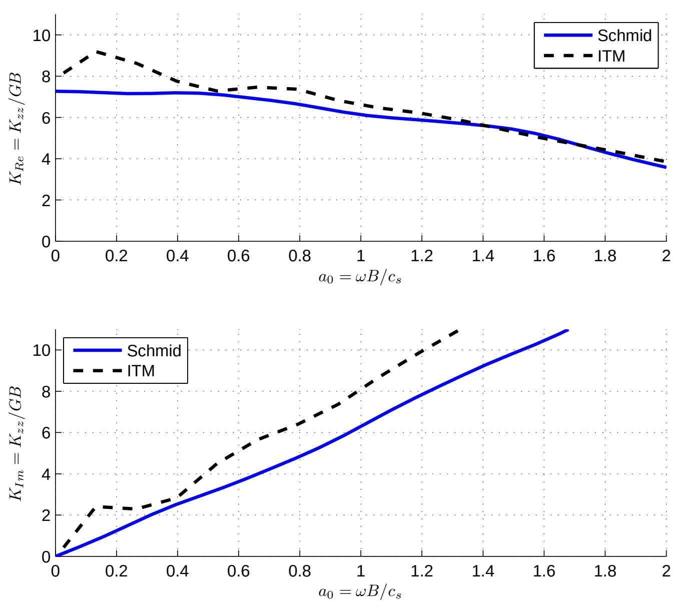


Figure 8. Vertical dynamic stiffness K_{zz}, $\nu = 0.4$

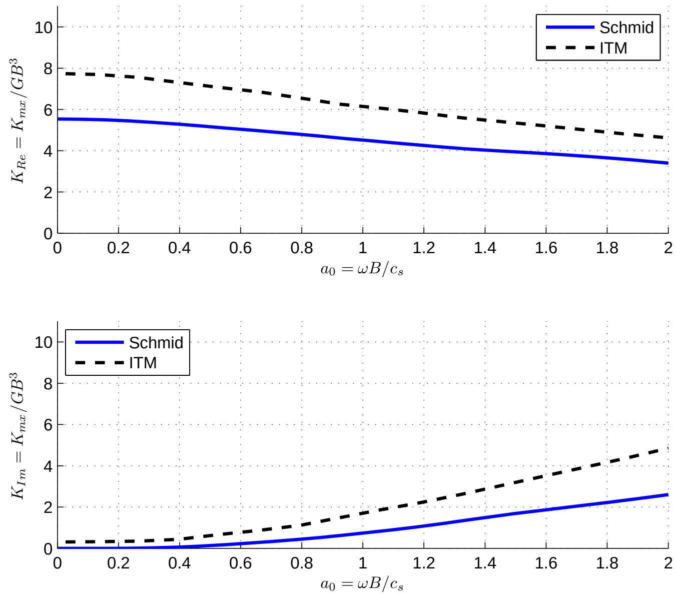


Figure 9. Rotational dynamic stiffness K_{mx}, $\nu = 0.4$

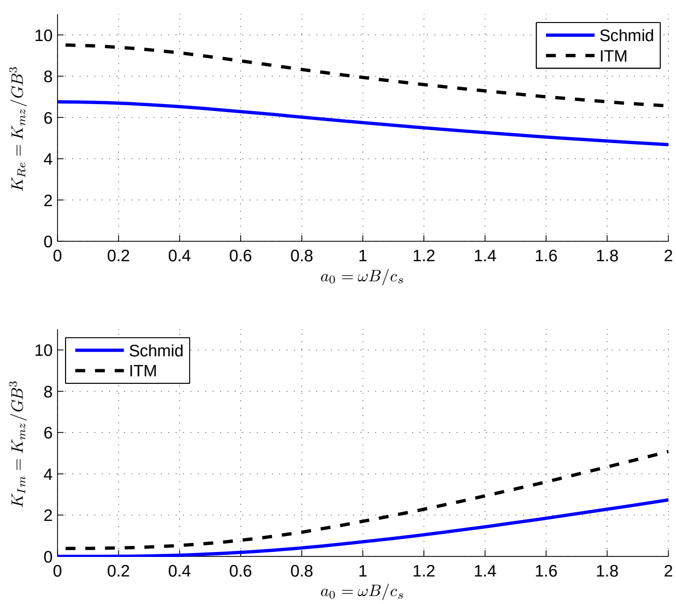


Figure 10. Torzional dynamic stiffness K_{mz}, $\nu = 0.4$

7. Conclusion

The *ITM* is used to calculate impedance functions for rectangular rigid foundation on a half-space. The obtained results show good agreement with results from literature. *Integral Transform Method* is based on the analytic solution of the wave propagation theory and transformed technique. The original problem is transfered to a new domain using *Fast Fourier Transform (FFT)*, where it can be solved much easily. The obtained results are returned into the original domain by *Inverse Fast Fourier Transform (IFFT)*. These transformations may demand a considerable computational effort.

ITM is restricted to the half-space and to a horizontally layered half-space, with a homogeneous and isotropic layers. In order to overcome this limitation for the case of local irregularities *ITM*-approach can be combined with *FEM*. Instead of *FFT*, Laplace transformation or *Wavelet transform* can be used.

The advantage of *ITM* is that damping is taken into account automatically, as material, or hysteretic damping as well radiation damping. The material damping is involved through complex modulus, while radiation damping is defined by *Sommerfeld*'s radiation condition.

This approach can be used for solving different problems of wave propagation in the soil, specially problems of rail or road traffic induced vibrations.

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