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- INVITED LECTURE -

GEOMETRIC PROPERTIES OF THE "FLOWER" CONCAVE ANTIPRISMS OF THE SECOND SORT

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ABSTRACT

This study presents a continuation of the research on the concave polyhedra of the second sort, adding to this family a new group of related polyhedra. They are formed over a specific type of isotoxal concave polygons that allow geometric arrangement of a double row of equilateral triangles into formations which enclose a deltahedral lateral surface without overlaps and gaps. As in all other representatives of the concave polyhedra of the second sort, we expect to find here the "major" and "minor" type, depending on the way we fold the net. This research has identified both these polyhedra types, which have the same planar net, but are formed over different basic concave polygons. The origination, constructive methods and properties of these solids are elaborated in the paper.

Keywords: concave polyhedron; concave polygon; polygon elevation; antiprism; isotoxal; constructive geometry

INTRODUCTION

Triangulated surfaces, as one of the most common ways of visualizing complex free-form shapes, as well as parametric and implicit surfaces are among the most recognizable features of computer graphics in recent decades. Numerous software programs use algorithms that can represent any shape by a series of connected triangles, so that irregular deltahedral surfaces, made up of arbitrary (scalene) triangles, form the weft of any free-form shape. Thus, triangular nets have the ability to obediently apply our ideas via digital tools and adapt to any desired surface, discretizing it. However, if we introduce more geometric regularities into the process of forming the net itself and also increase the level of the regularities expected from the resulting shapes, the free form inevitably gives way to the geometric one.

If we introduce a requirement that each triangle should be equilateral, the surface continuous and convergent, we will pose a problem that can be solved by a different, perhaps simpler approach. Knowing the properties and relations of linear and angular measures within an equilateral triangle, we can solve the task with classical geometric methods, from trigonometry to constructive geometry. Given an additional condition: that these regular deltahedral surfaces should also encompass regular polygonal bases, we come to a few convex solutions, such as Platonic solids: tetrahedron, octahedron and icosahedron, and Johnson solids: J12 and J13 (Johnson, 1966) while convexity alone adds three Johnson solids more: J17, J51 and J84. However, if we exclude convexity as a criterion, we are left with various other possibilities. Some of them have been described in a series of papers (Obradović et al.., 2013-II; Mišić et al.., 2015, Obradović et al.., 2017) dealing with composite concave deltahedra based on the geometry of concave polyhedra of the second (or higher) sort with regular polygonal bases. The later solids, the concave polyhedra of the second sort, serve as a foundation of this research as well. They include: concave cupolae (Obradović et al.., 2008), concave pyramids (Obradović et al.., 2014; Obradović et al.., 2015) and concave antiprisms of the second sort (Obradović et al.., 2013-I). The sort is determined by the number of rows of equilateral triangles in the lateral surface, which is directly related to the number of resulting solutions (ergo, heights of those solutions). The 'sorts' are regularly of an even number (Mišić, 2013), so we have concave polyhedra of the second, fourth, sixth, etc. sorts, except for the antiprisms which are always of the second sort.

In this paper, as an addition to the above concave polyhedra formed over regular *convex* polygonal bases, we look for concave polyhedra that are formed in a similar way, but with *concave* polygon of multilateral *k*-fold symmetry as the basic one. We start from the ones that can create an infinite deltahedron (Wachman et al.., 1974). Those are (concave) antiprisms, formed over two identical polygonal bases. Given that their star-like bases are not actually stellations of convex polygons, the solutions obtained will not be polygrammatic antiprisms (Huybers, 2001), but original solids with all edges of equal length and with deltahedral lateral surfaces. This paper explores the shapes of these bases and the ways to form the solids out of the same triangular net so that we get feasible solutions without gaps or overlaps of the triangular faces.

2. RESULTS OF PREVIOUS RESEARCH

Two previous studies are crucial to understanding the method and the results of this research. The first one (Obradović et al.., 2013-I) concerns the definition of the forms and measures of the concave antiprism of the second sort, CA-II. We have shown that they are formed, similarly to the concave cupolae or concave pyramids of the second sort, by folding a net of double series of equilateral triangles, and that, depending on the way the net is folded, they may be higher (major, M) or lower (minor, m) in height. In other words, they can exist as two types of the concave antiprisms, denoted by: CA-II-nM and CA-II-nm. Fragments of these solids' surface were used to obtain concave rings-like surfaces, crucial for the next study.

The second study (Obradović et al.., 2019) considers a special way of assembling the major type of the solids mentioned above, CA-II-nM, so that their lateral surfaces form deltahedral closed rings. These rings are consequently of the second sort, thus denoted by CDR-II. They are formed by closing the full circle via 2π polar array of unit *spatial decahedral cells*, shown in Fig. 1 a) and b), formed by two partly overlapped unit hexahedral cells of two adjacent CA-II-nMs (Obradović et al.., 2019, Fig. 2). To perform a precise array, we have determined the angle $\boldsymbol{\omega}$ (or $\boldsymbol{\varphi}$) found between the symmetry planes (σ or τ) of the adjacent pair of CA-II-nMs (Fig 1 c and d). The planes, seen as the axes of symmetry in the top orthogonal view are defined by the outermost points of the unit decahedral cells. Their multiplication by an integer $\boldsymbol{K}$ gives a full circle, so that the ring is obtained.

In this paper, only the angle $\boldsymbol{\omega}$ is relevant for the calculus and the construction of the lateral surface.

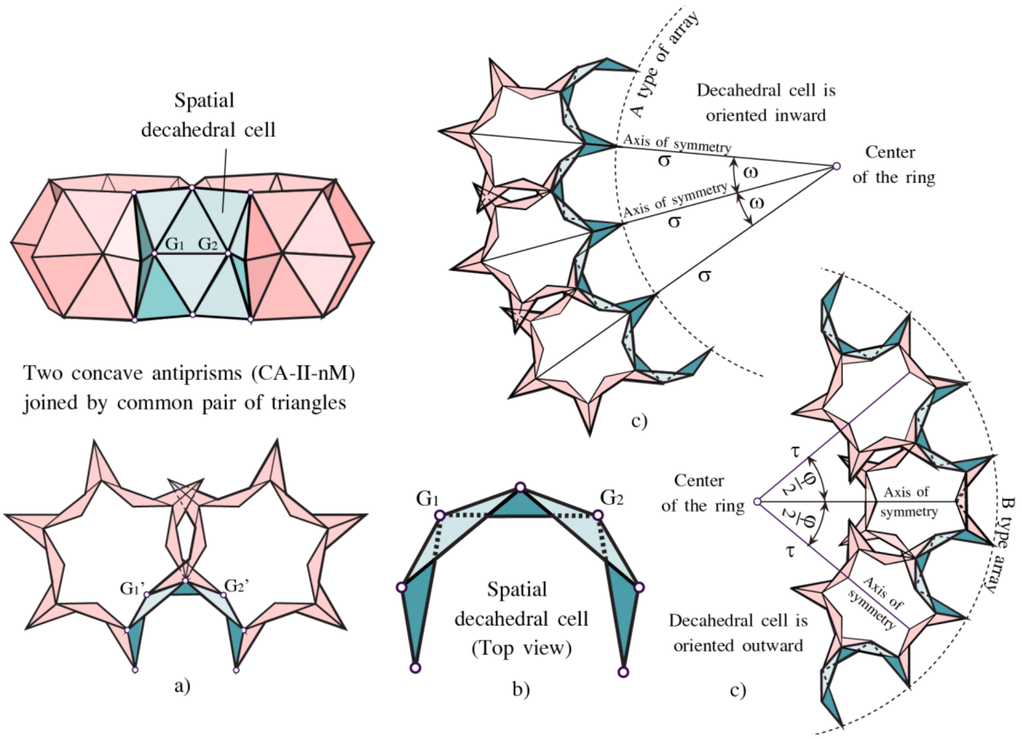

Figure 1: (a) Two paired concave antiprisms of the second sort (CA-II-nM) (b) Unit spatial decahedral cell (c) Two ways of forming concave deltahedral rings (CDR-II) using the unit decahedral cells (Source: Obradović et al. (2019); illustration is modified Fig. 2)

The obtained rings share key features of concave polyhedra of the second sort: multilateral *k*-fold symmetry, deltahedral lateral surface and the same folding methods, which produce its major and minor type. Naturally, since they are derived from CA-II-nM's fragments, they share their linear and angular dimensions.

In the top view, the shapes of CDR-II can be flower-like or star-like, with distinct shapes (A, B) or even more complex, mixed with fragments of CA-II-nMs (A_f, B_f) in between "petals" or "star-points", depending on the number of sides (n) in the CA-II-nM's base polygon from which the rings originate. The number n conditions the number k, which represents the number of decahedral cells in the ring. If the full ring is closed with k unit cells, then $k=K$, which is the number of “petals” or “star points” in the newly obtained rings. Based on the formulae (1) and (3) in (Obradović et al.., 2019), for the angle $\boldsymbol{\omega}$, and having that: $k_{A,B} = \frac{2\pi}{\omega}$ the interdependence between n and k can be expressed as:

$$\boldsymbol{k} = \frac{3n}{n-5};\ \boldsymbol{n} \in \mathbb{N} \qquad \text{(Eq.1)}$$

The aforementioned study, of which this research is a continuation, considered only the forms of rings obtained by using (decahedral) fragments of CA-II-nM, without dealing with geometry of the bases thus obtained, since eight different possible ring shapes can be identified: A, A_f, A_a, B, B_f, B_{f3}, C and C_f. Thus, n had to be an integer, as the number of sides in the basic regular polygon of the CA-II-nM, while k was usually a rational number, and in only a few special cases an integer too (when the pure forms A and B of the ring are obtained). Therefore, in general case, we needed to correct k by a factor j to convert the fraction k to an integer K, which represents the number of “petals” or “star points” in the deltahedral rings.

In this study, the subject of the research are solids whose lateral surfaces are just pure forms (A and B) of concave deltahedral rings, with a number of "petals" that can be of any integer $k \in \mathbb{N}$ (CDR-II-k). Also, unlike the previous research, we do not examine only the lateral surfaces, but the whole solid, including the shapes of the base polygons. In addition, we are looking for a connection between the A and B ring shapes and the major and minor solid types.

3. INITIAL SETTINGS OF THE RESEARCH

This paper investigates an entire group of polyhedra generated with a precise polar array of k spatial decahedral cells, with no additional cells in between. Hence, we are looking for the distinct forms of "flowers" only, which are obtained by folding the net of the deltahedral lateral surface. In this case, the number n will not always be an integer as it was with CDR-II, but the number k will, as an assigned number of sides in the regular k-sided polygon which we elevate[1] (Grünbaum et al.., 1986) by triangles, i.e. "petals" of the new non-convex, $2k$-sided polygon. This $2k$-sided, star-like isotoxal[2] polygon will be the base polygon of the “flower” concave antiprism, and finding its geometrical properties is one of the foci of this paper.

We are investigating a special type of concave polygons that arise as elevated k-sided regular convex polygons, and have the property that their concave angles are conjugate to the interior angles of the n-sided convex polygon.

3.1. Minor Type of Floral antiprisms of the second sort

Although in the previous research we used CA-II-nM, i.e. the antiprisms of the **major** height for the procedure of obtaining CDR-II-K, the resulting structures came out to have all the characteristics of the lower-height solid type. Why? Because the disposition of protruding and indented vertices and edges, when viewed from the exterior, gives the result that corresponds to the way the triangular net is folded to get the minor type of concave polyhedra of the second sort. The central vertices G_1 and G_2 of the unit decahedral cell (Fig. 1a, b), originating from the central vertex G of the hexahedral cell in the CA-II-nM are protruding, which gives a lower height (Obradović et al.., 2013-I), as it is the case with all the other C-II-n representatives (Obradović et al.., 2008; Obradović et al.., 2014; Obradović et al.., 2015). Given that we are here applying the same method of folding the triangular net in two different ways, major and minor, with this we obtain the lateral surface of minor height, which will be evidenced by the elaboration of the major type, discussed later.

Let us first look at the geometric properties and the method of generating the “flower” antiprisms of the second sort with k petals, of minor type (FA-II-km). Inverting the procedure of the previous research, now the input is k - the number of "petals" we wish to obtain, so the number n - the assumed number of the sides in CA-II-nM that produces this deltahedral surface is derived from the above formula (Eq. 1):

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[1] Grünbaum used the term „elevatum“ for the positive height, i.e. pointing outward pyramids in augmentations (Weisstein https://mathworld.wolfram.com/Augmentation.html) .

[2] Isotoxal polygon is an equilateral, edge transitive polygon, i.e. it has only one type of edge.

$$n_m = \frac{5k}{k-3};\ k \in \mathbb{N} \tag{Eq.2}$$

Although n will not be an integer in most cases, this will not interfere with the formation of deltahedral ring (CDR-II-k) for any $k \in \mathbb{N}$. In all cases, the interior angles of n-sided polygon define the exterior angles of the star-like $2k$-sided polygon.

In Fig. 2 we see the starting setup with the elements we use to trigonometrically determine all the necessary angular measures for defining the basis $2k$-sided polygon.

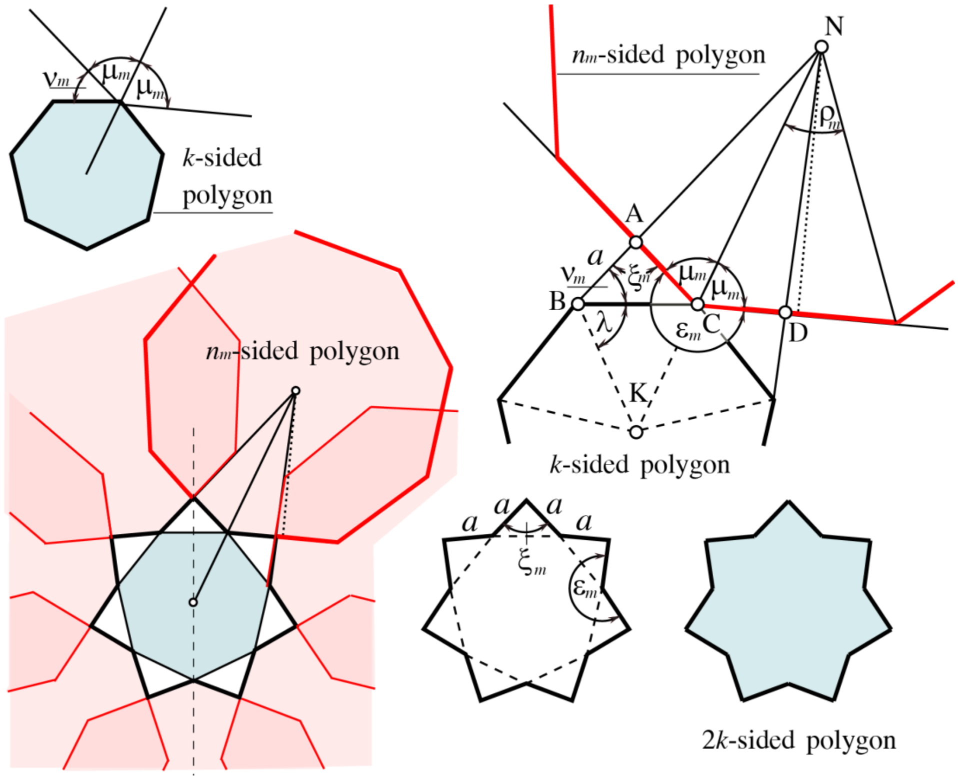


Figure 2: The way of forming the basic 2k-sided star-like polygon based on the k-sided regular convex polygon, the key angles and the final shape of the polygon (the example of heptagon)

We first need to determine the value of the angle μ_m, the interior angle of the n-sided polygon, then to define the other necessary angles depending on it, above all the angle ε_m in the "star-polygon" as its conjugate angle, and the angle υ_m as the base angle of the elevatum triangle. This way, a special type of concave, isotoxal polygon arises as an elevation of the k-sided regular convex polygon.

The calculated values of the angles: ρ_m, λ, μ_m, υ_m, ξ_m and ε_m (Fig. 2) relevant for determining the interdependence between the n-sided and $2k$-sided polygons are given by the formulas in Table 1.

Table 1: Overview of all the angles relevant for determining the 2k-sided star-like concave polygon (base "flower" polygon) for the minor type of "flower antiprism" FA-II-km

Angle	=	$f(k)$	*Equation*
ρ_m	=	$\frac{2\pi(k-3)}{5k}$	(Eq.3)
λ	=	$\frac{\pi(k-2)}{2k}$	(Eq.4)
μ_m	=	$\frac{3\pi(k+2)}{10k}$	(Eq.5)
υ_m	=	$\frac{\pi(k+2)}{5k}$	(Eq.6)
ξ_m	=	$\frac{\pi(3k-4)}{10k}$	(Eq.7)

ε_m	=	$\frac{\pi(7k-6)}{5k}$	(Eq.8)

The obtained polygon will be neither a regular star polygon[3] (Kepler, 1619/1968; Coxeter, 1969) i.e. stellation, or polygram, nor a regular compound polygon (see: Bowers, 2012), except in two special cases. It is an equilateral, isotoxal star polygon with angles such that:

$$\varepsilon_m = \frac{\pi}{2} + 3\xi_m \quad \text{(Eq.9)}$$

Table 2 shows the interdependencies of the numbers k and n. We can see how the values of n in the outer polygon change as the values of k increase. We notice that when k tends to infinity, n tends to number 5, which can also be verified by the Eq. 1.

Table 2: Minor type of FA-II-k; Interdependence of the number of sides of two conjugated polygons: *k*-sided and *n*-sided

k	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
n_m	∞	20	12.5	10	8.75	8	7.5	7.14	6.875	6.66	6.5	6.36	6.25	6.15	6.07	6	5.94	5.88	5.94	5.88

We find that both k and n are integers only in 4 cases (plus 2 in which the associated number is infinite), as shown in Table 3. This, however, does not in any way interfere with the formation of the star polygon for any value of $k \in \mathbb{N}$. These 4 cases are the ones where CA-II-nM and FA-II-km can be adjoined and match face to face so that they can be arranged as modular elements.

Table 3: Minor type of FA-II-k - Cases when both k and n are integers

k	3	4	6	8	18	∞
$n_m = \frac{5k}{k-3}$	∞	20	10	8	6	5

In the case of $k = 8$, there is a unique situation when the *2k*-sided polygon is simultaneously a stellated polygon - obtuse octagram {8/2} (Grünbaum, et al.., 1986), a regular compound polygon and a flower (minor) polygon.

As stated above, the sides of this “flower” star polygon are all of equal length, *a*, which is the side of an equilateral triangle in the lateral surface. The question we have to resolve now is what the arrangement of equilateral triangles in the deltahedral surface of such an FA-II-km will look like.

3.2. Description of the Constructive Procedure

Each “petal” of the “flower” antiprism consists of ten equilateral triangles, organized as described in (Obradović et al.., 2019) and forming a unit decahedral cell (see Fig. 1). These cells are connected to the adjacent ones by the outermost edges and, thus arrayed, they close the annular lateral surface.

The decahedral unit cell has two planes of symmetry:

- the vertical one, passing through the midpoint of the edge G_1G_2 and the centroid (K) of the base polygon,
- the horizontal one, passing through the vertices G_1 and G_2, parallel to the planes of the bases.

Due to the later, in the top view, the triangles that constitute it are seen as overlapped, two by two. Thus, we will see only 5 triangles organized around each “star elevatum” ABC of the initial *k*-sided polygon, the matrix for the *2k*-sided isotoxal "flower" polygon (Fig. 3 c).

It is imperative that in the first projection (top view) all the triangles from the lateral surface should be projected into congruent triangles, since both bases of the concave antiprism are, by definition, congruent and all the

[3] According to one of Kepler's definitions, not only stelations, but isotoxal concave polygons are also named 'star polygons'.

triangles are set at the same angle with respect to them, in order to span the same height (Fig 3 b). Therefore, the base angles in these isosceles triangles are all equal and amount to the value denoted by α.

The 5 visible triangles (seen in top view) from the unit decahedral cell are arrayed in an open polygon with 7 vertices: H_1, C, G_1, A, G_2, B and H_2 (enumerated clockwise) as shown in Fig 3 a) and b). Given that 3 of these 7 vertices coincide with the vertices ABC of the base star polygon, we come to the conclusion that all the (projected) vertices of the decahedral unit cell lie on the same circle *c*, determined by the vertices A, B, and C. The vertices G_1 and G_2 are easily found in the intersection of the normals of the sides AB and BC respectively, with the circle *c* itself (Fig. 3 a), while the vertices H_1 and H_2 are found at the intersection points of the circle *c* and the symmetry axes - planes (σ) of the base polygon, set through the vertices B and C respectively.

Then, with the polar array of *k* identical decahedral cells, and with the center in the intersection point (K) of their symmetry planes (σ), we form the entire ring (Fig. 3 d).

In Fig. 3 e) we can see the first orthogonal projection of the "flower" antiprism's lateral surface thus obtained.

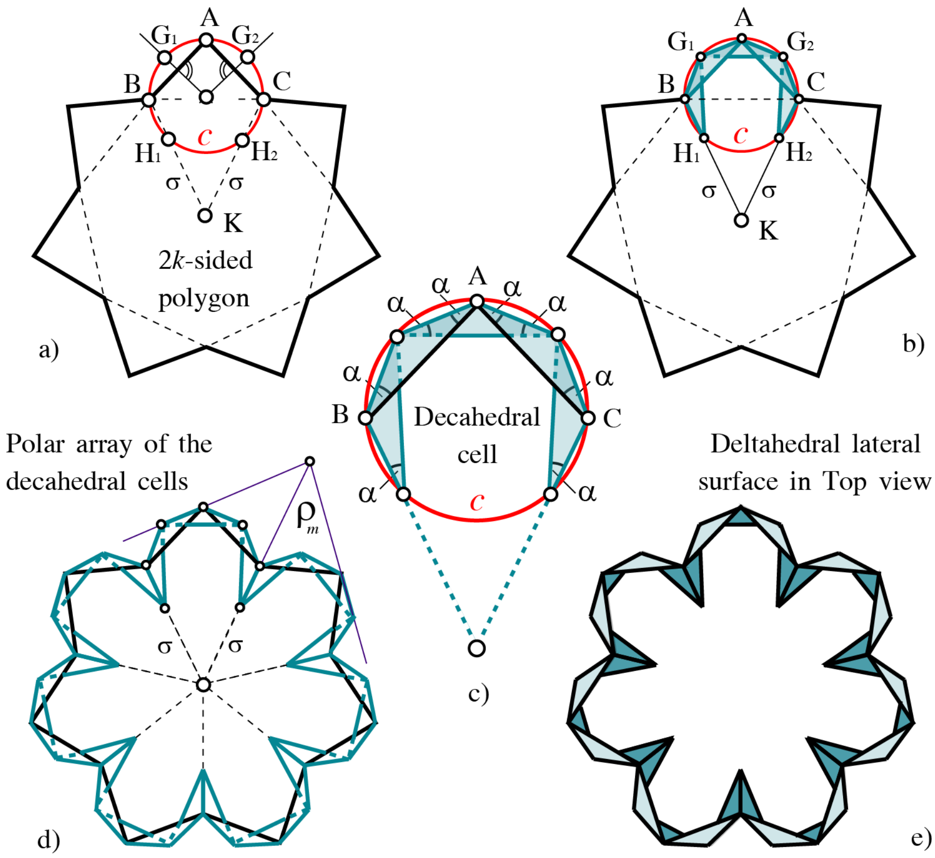


Figure 3. Construction and polar array of a single decahedral cell

3.3. Determining the height of the solid

In order to fully define the solid, we need to determine the heights of the vertices. According to formula (1) in (Obradović et al.., 2019), the angles α seen in the first projection are the base angles in the isosceles triangle, which every triangle from the deltahedral surface is projected into. We can now calculate this angle α as $f(k)$.

$$\alpha_m = \frac{\pi(k+2)}{10k} \quad \text{(Eq.10)}$$

Based on this angle, we can calculate the height of the solid itself. On the other hand, we can easily obtain it via constructive procedures based on the 3D transformations of an equilateral triangle (Fig. 4 a).

$$h = \frac{a}{2}\sqrt{3-(tg\alpha)^2} \quad \text{(Eq.11)}$$

The constructive - geometric procedures for obtaining the height of the "flower" antiprism (h_{FA}), i.e. the heights of the vertices A, B and C, start from the assumption that the vertices G_1, G_2, H_1 and H_2 lie in the horizontal plane at height h=0. Two of them are shown in Figure 4. We can use:

- the method of introducing a new transformation plane (2') in the orthogonal view, given in Fig. 4 a)
- the method of 3D rotation of the equilateral triangle's edge (H_1B) in a 3D environment, given in Fig. 4 b).

With the obtained heights of the vertices A, B and C we easily form lateral triangles (Fig. 4 b), and then the whole lateral surface, considering that the entire structure should be repeated, i.e. "mirrored", with respect to the horizontal plane of symmetry (Fig. 4 c).

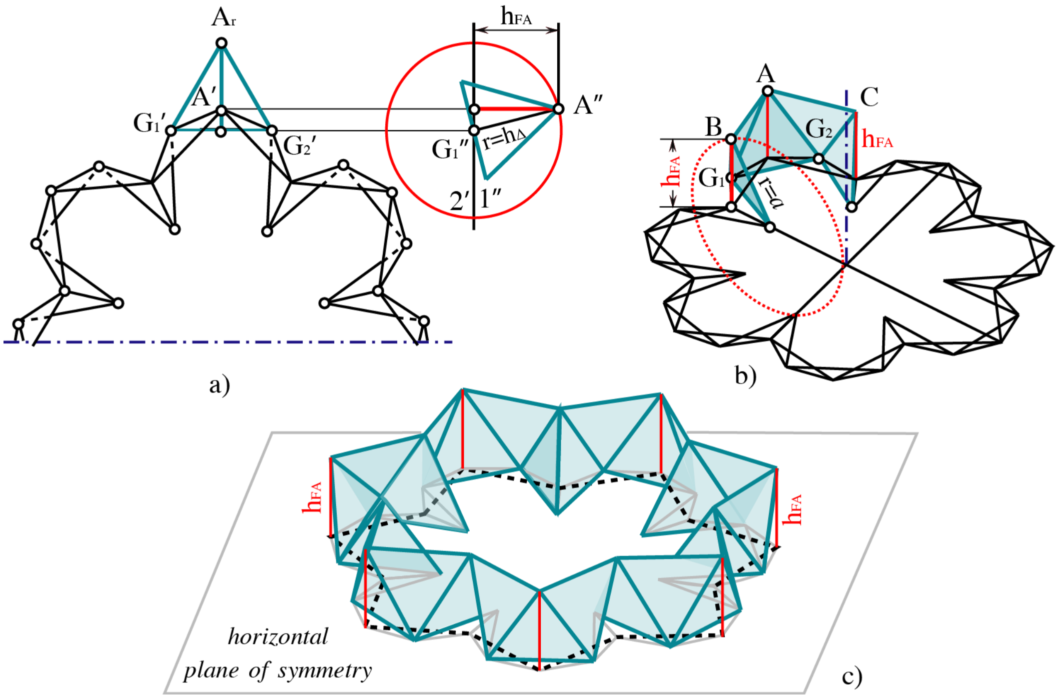


Figure 4: Construction of the vertex heights: (a) transformation plane method, (b) 3D rotation method (c) half of the lateral surface

With this constructive procedure we can create a "flower" antiprism for any *k*. Examples of several "flower" antiprisms of minor type (FA-II-km), for initial $k \in \{3, 4, 5, 6, 7, 8\}$ are shown in Fig. 5. We can observe the outlines of the *2k*-sided "flower" polygons in top view, as well as the models of *k*-merous "flower" antiprisms themselves in axonometric view.

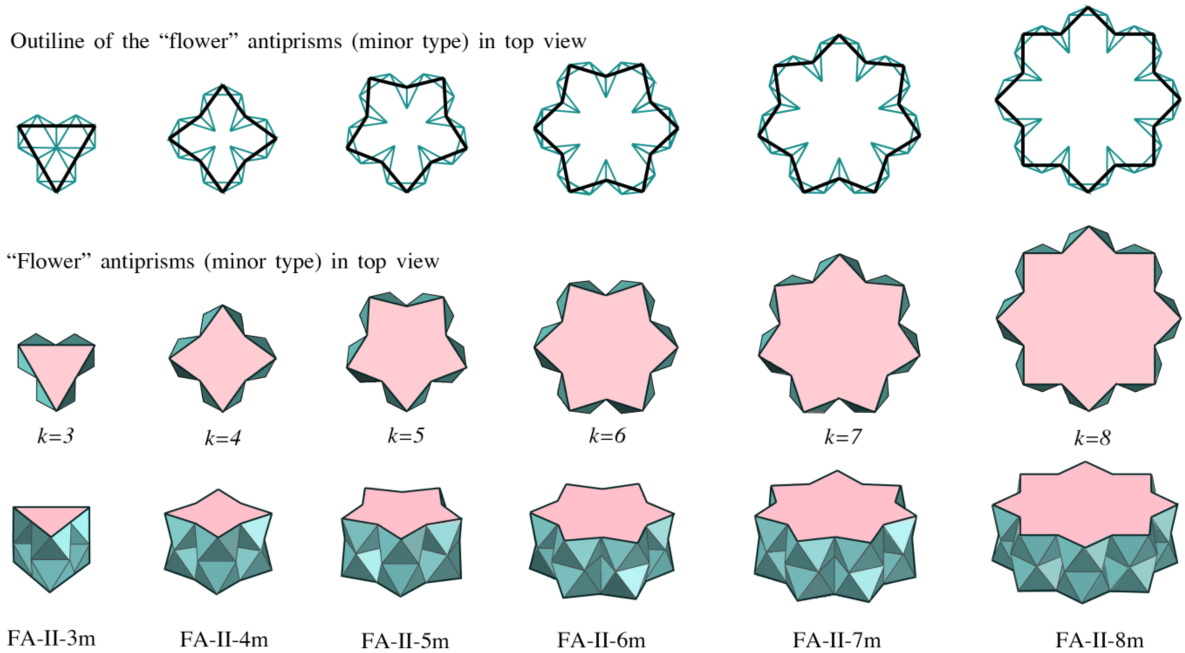
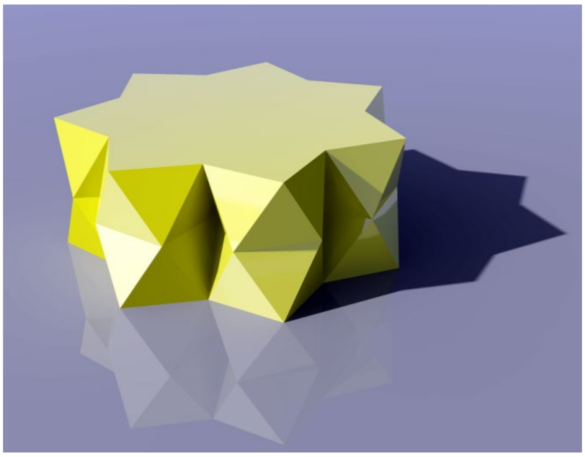


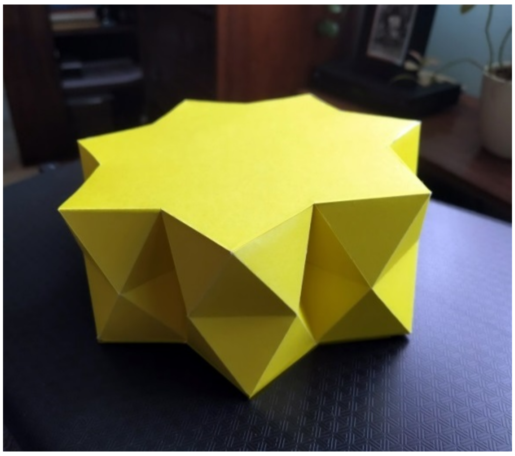
Figure 5: Some representatives of "flower" antiprisms, with $k \in \{3, 4, 5, 6, 7, 8\}$

All the solids shown are verified through 3D models performed in AutoCAD application. In Fig. 6 a) we see FA-II-7m as a rendered image of a 3D model, while in Fig. 6 b) we see a photograph of a physical model of the same solid.

In this way, we prove the accuracy and feasibility of these forms.



(a)



(b)

Figure 6. An example of FA-II-7m: (a) Rendered image of a 3D model, (b) Photograph of a physical model

4. MAJOR TYPE

As stated above, and according to all the previous research on concave polyhedra of the second sort (Obradović et al.., 2008; Obradović et al.., 2013-I; Obradović et al.., 2014; Obradović et al.., 2015), two types of lateral surface, major and minor, can be formed from a single net of double-rowed equilateral triangles, depending on the folding mode. The situation here will be similar to that described above. From a triangular grid, we separate a segment corresponding to k connected decahedral cells and thus again obtain $10 \cdot k$ equilateral triangles organized in two rows. The net of the 7-merous "flower" antiprism is given in Fig.7. By folding it in the manner that the vertices G_1 and G_2 of the unit decahedral cell are protruding into exterior, we get a minor type of "flower" antiprism, described in the previous.

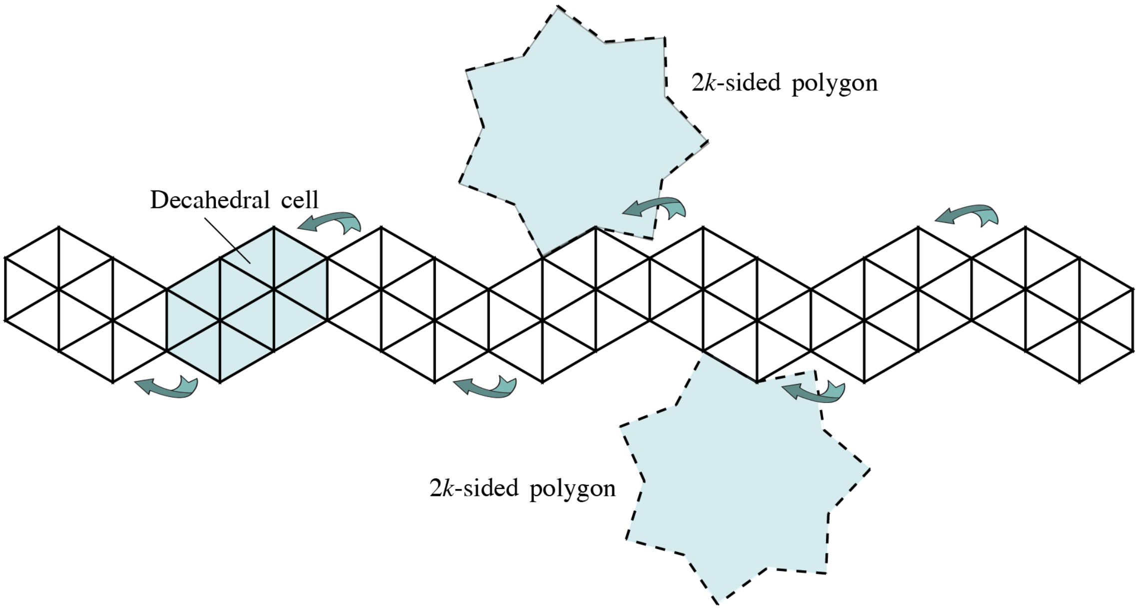


Fig. 7: Net of the "flower" antiprism

Now, to get the major type, we fold the net in the alternative way so that the vertices G_1 and G_2 become indented, as viewed from the outside.

However, if we keep the same base polygon, we will find that it is not possible to obtain the equilateral deltahedral surface, due to constructive and trigonometric paradoxes. Let us look further into this.

From Fig. 8 we can see that angle χ, viewed as angle NH_1A in the quadrilateral (deltoid) NH_1AG_1, is:

$$\chi_1 = \frac{2\pi-\rho}{4} = \frac{\pi(4k+3)}{10k} \qquad \text{(Eq.12)}$$

At the same time, this angle must satisfy the condition from the quadrilateral NH_1QC, that as the angle NH_1Q it is:

$$\chi_2 = 2\pi - \frac{3\rho_m}{2} - \upsilon_m - \mu_m - \frac{\pi}{2} = \frac{2\pi(k+2)}{5k} \qquad \text{(Eq.13)}$$

These two values, χ_1 and χ_2 of the angle χ, cannot be equal for any value of k, (Fig. 8 a) if the values of ρ_m, μ_m and υ_m are expressed as functions of n_m, because its relation to k is defined by Eq. 2, for the minor type. The given relation does not simultaneously satisfy the setting for the major type. Therefore we must look for other value, n_M, and consequently all the related angles, to satisfy the above condition.

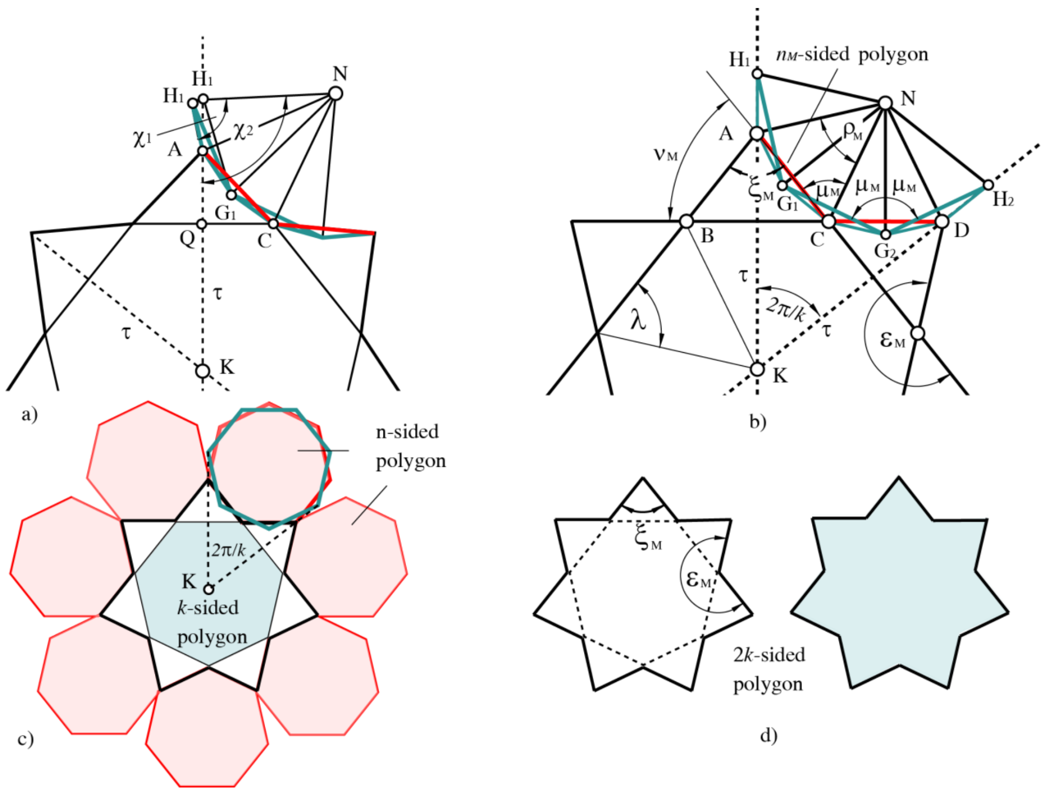


Figure 8. Starting elements for determination of FA-II-kM

Although it will not be feasible to use the same “minor” polygonal basis, we can still assemble a major type of lateral surface out of the same net, with the same number of “petals”, and with the central vertices G_1 and G_2 being indented. Accordingly, the basic star polygon will be slightly different.

The outmost edges (AH_1 and DH_2) of the decahedral cells must lie on the planes of symmetry (τ) of the basic star polygon, and at the same time, the vertices of the lateral triangles (H_1, A, G_1, C, G_2, D and H_2), projected into vertices of the isosceles triangles, must form a *2n*-sided polygon, whose 6 consecutive sides form the contour of the major "flower" antiprism (Fig. 8 b). This means that the n_M-sided polygon is determined by the angle between the planes τ, which is $2\pi/k$.

The value of n_M is given by Eq. 14, and the value of the base angle α of the projected isosceles triangle in Eq. 15.

$$n_M = \frac{5k}{k-2};\ k \in \mathbb{N} \qquad \text{(Eq.14)}$$

$$\alpha_m = \frac{\pi(k-3)}{10k} \qquad \text{(Eq.15)}$$

If we apply rotational symmetry on the n_M-sided polygon by the angle $2\pi/k$ and with the center of the rotation in the centroid K of the k-sided base polygon, the new arrangement of n_M-sided polygons accurately plot the sides, “petals” of the basic *2k*-sided polygon (Fig. 8 c).

The major-type of the *2k*-sided polygon, now with a different n_M-sided conjugate polygon, obtains its final appearance, given in Fig. 8 d.

The calculated values of the angles: ρ_M, λ, μ_M, υ_M, ξ_M and ε_M (Fig. 8 b) relevant for determining the n_M-sided and *2k*-sided polygons are given by the formulas in Table 4.

Table 4: Overview of all the angles relevant for determining the *2k*-sided star-like polygon (base "flower" polygon) for the major type of “flower antiprism” FA-II-kM

Angle	=	$f(k)$	*Equation*
ρ_M	=	$\frac{2\pi(k-2)}{5k}$	(Eq.16)
λ	=	$\frac{\pi(k-2)}{2k}$	(Eq.17)
μ_M	=	$\frac{\pi(2k+1)}{5k}$	(Eq.18)
υ_M	=	$\frac{\pi(k+3)}{5k}$	(Eq.19)
ξ_M	=	$\frac{3\pi(k-2)}{5k}$	(Eq.20)
ε_M	=	$\frac{\pi(7k-4)}{5k}$	(Eq.21)

Table 5 shows the relationship between numbers of sides in k-sided and n_M-sided polygons, if we assign k to be an integer.

Table 5: Major type of FA-II-k; Interdependence of the number of sides between two conjugated polygons: *k*-sided and *n*-sided

k	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
n_M	**15**	10	8.33	7.5	7	6.67	6.43	6.25	6.11	6	5.91	5.83	5.77	5.71	5.67	5.62	5.59	5.55	5.59	5.55

Only in four cases are both k and n_M integers, as it is shown in Table 6. There we see that if k tends to infinity, n_M tends to number 5, as for the minor type of “flower” antiprism, but now there are no cases when n_M is infinite, since the condition for that is *k*=2, which is not possible for a polygon.

Table 6: Major type of FA-II-k – Cases when both k and n are integers

k	2(-)	3	4	7	12	∞
$n_M = \frac{5k}{k-2}$	∞	15	10	7	6	5

Once again, just like for the minor type, in the first projection (top view) all the triangles from the lateral surface are projected into congruent triangles (Fig 9 c). In this case also 3 out of 7 visible vertices of the unit cell coincide with the vertices A, C and D of the “flower” polygon, i.e. belong to both k-sided and n_M-sided polygons (Fig 8 b and Fig 9 a). The vertices G_1 and G_2 can also be found in the intersection of the normals of the sides AC and CD respectively with the circle *c* itself, while the vertices H_1 and H_2 are found at the intersection points of the circle ***c*** and the symmetry plane (τ) of the base polygon, set through the vertices A and D respectively (Fig 9 a). The remaining 4 vertices belong to an identical but gyrated polygon to the n_M-sided one, so one more time all the 7 vertices of the unit cell, seen in the top view, will lie on the same circle *c*, determined by the vertices A, C, and D (Fig. 9 b).

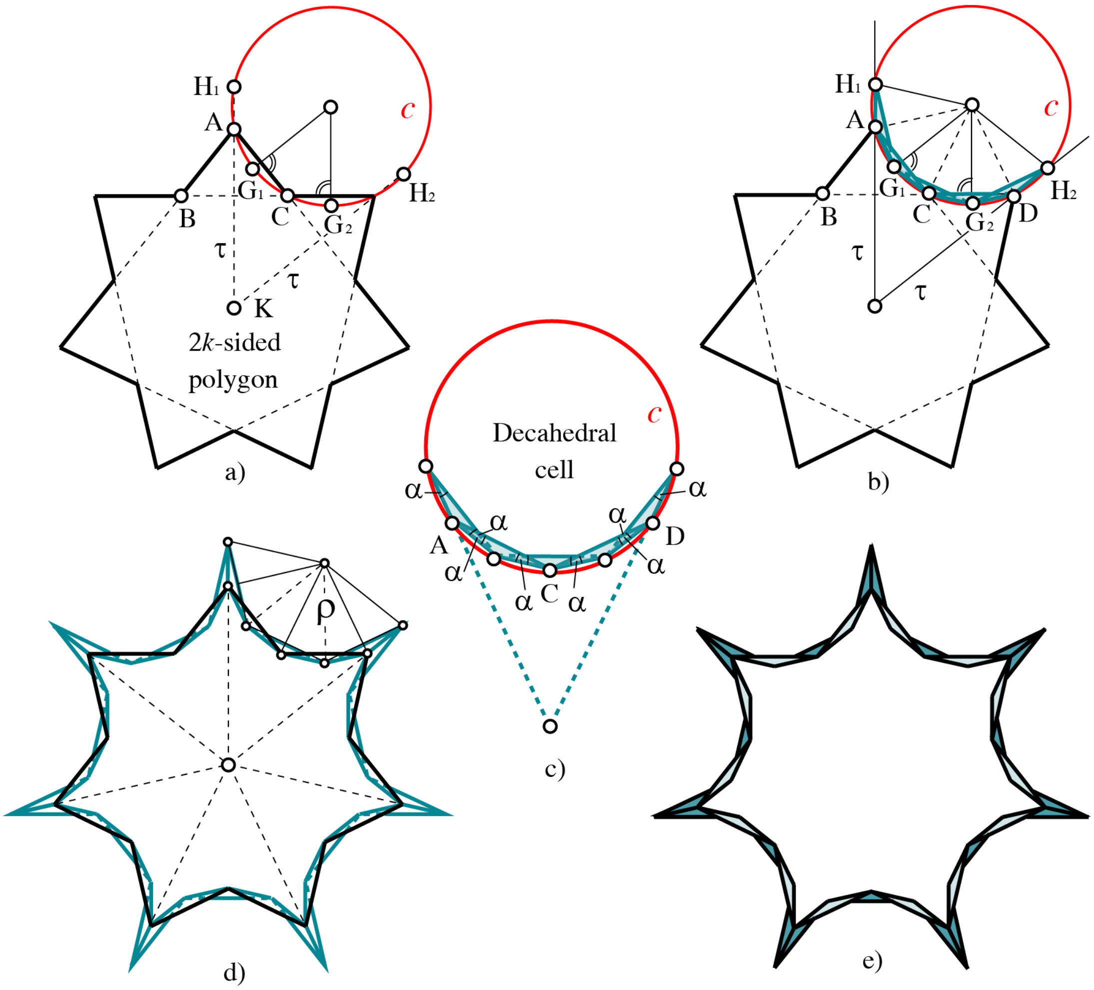


Fig 9: The construction and origination of "flower" antiprism of the major type

Accordingly, another way to create the contour of a "flower" antiprism's unit cell of the major type is to have the polygon n_M girated. Then, with its polar array, we form the entire ring, observed in top view (Fig 9 d). In Fig. 9 e) we can see in the final contour outlook of the lateral surface with the obtained *k* (7) "petals" of the "flower" antiprism, also in the top orthogonal view.

Next we proceed to determination of vertices' heights in order to obtain all the necessary elements for construction of the solid. We apply the same methods as for finding the height of the minor type, described in the section 3.3.

It is interesting to note that the minor and major types differ not only in height and in the way the net is folded and assembled, but also in two additional interesting details:

- the area of the major polygon is larger than that of the minor polygon,
- surfaces whose fragments form the lateral surface of the major type of the "flower" antiprism, with the equivalent role that CA-II-nms had in the formation of the minor type of the "flower" antiprism, are nothing else than double convex antiprisms. In this way, the connection between convex and concave antiprisms is confirmed again, as shown by Obradović et al (2019), but in the inverse procedure, where double convex antiprisms occurred as a consequence of the polar arrangement of CA-II-nm fragments.

Also, we will notice another link between the present and the previous studies: the case B of the CDR-II (illustrated only by the example with fragments of CA-II-4ms) described in Obradović et al (2019) is nothing but the lateral surface of the major type of "flower" antiprism, and the case A of those CDR-II-Ks is in fact the lateral surface of the minor type.

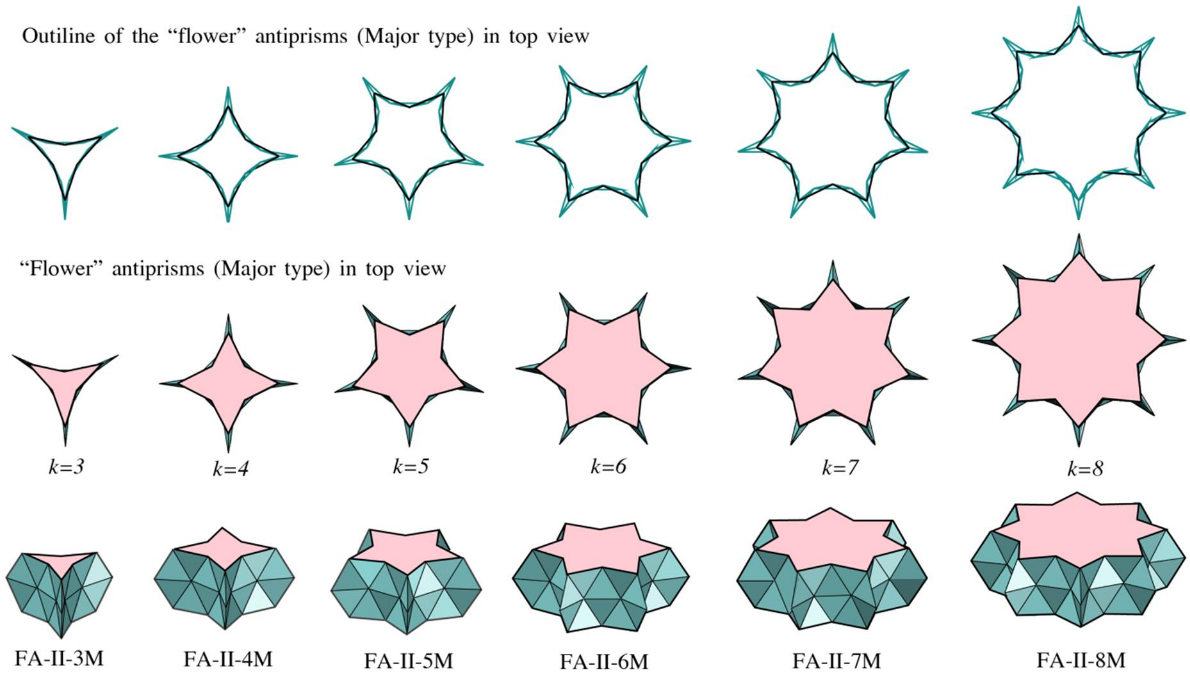
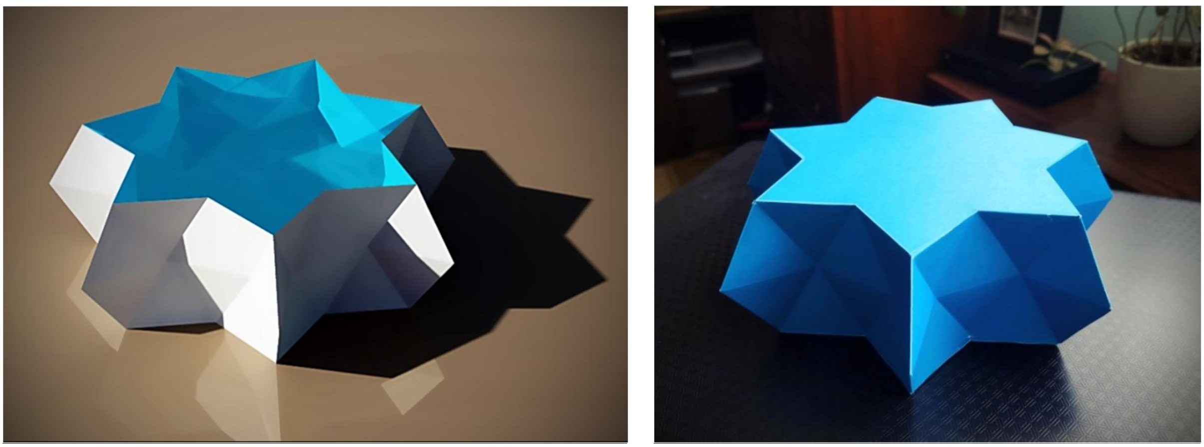


Figure 10. Some representatives of “flower” antiprisms, major type, with k=3 to k=8

Examples of several “flower” antiprisms of major type (FA-II-kM), for initial $k \in \{3, 4, 5, 6, 7, 8\}$ are shown in Fig. 10. We can observe the outlines of the *2k*-sided "flower" polygons in top view, as well as the *k*-merous "flower" antiprisms themselves in top and axonometric views.

In Fig. 11 a) we see a rendered image of a 3D model of the FA-II-7M performed using AutoCAD application, while in Fig. 11b) a photograph of the same solid’s physical model is presented.



a) b)

Fig. 11: a) Rendered the 3D model of the FA-II-7M; b) photograph of the physical model of the FA-II-7M

Finally, comparing the basic “flower” polygons of the FA-II-km and FA-II-kM representatives, with the same initial *k*-sided polygon (Fig. 12), we can observe that although they are not congruent, the divergence between major and minor polygons decreases with increasing the number *k*. The minor type of the polygon is given in red, and the major is given in black. The polygons equalize when *n=5*, while *k* is tending to infinity.

Note: If we compared these polygons for the same triangle edge *a*, the differences would be even less noticeable. Also, we will find that stellar polygons (polygrams) appear in two special cases only: for *k=8* in the minor case, where obtuse octagram {8/2} (Grünbaum, et al.., 1986) appears, and for *k=7* in the major case, where obtuse heptagram {7/2} is obtained.

In this way, the properties and measures of "flower" antiprisms of the second sort are completely defined.

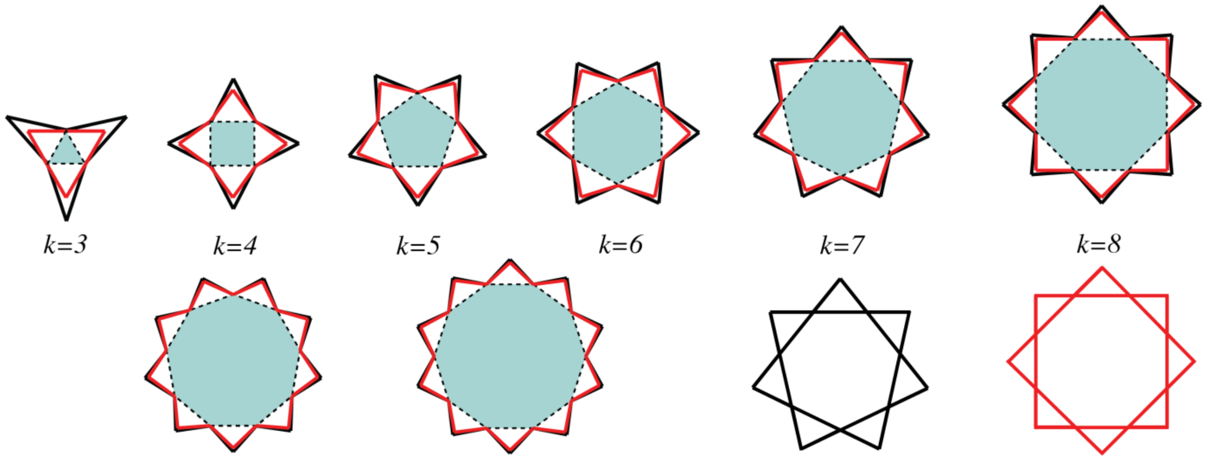


Fig. 12: Comparison of Major and minor representatives of FA-II-k with two special cases (k=7 for M, and k=8 for m) - polygrams

We can use these polyhedral surfaces to form infinite polyhedra, more precisely: infinite deltahedra, but only of the same type FA-II-k, without combining the major and minor types, as it was possible with CA-II-ns.

CONCLUSIONS

Based on all the above presented in the paper, the following conclusions are drawn:

- By using unit decahedral cells made of equilateral triangles, analogously to those formed by the use of CA-II-nM’s lateral surface fragments, we can take any integer *k* of them to form a closed deltahedral surface with multilateral *k*-fold symmetry.
- We can form a solid, a "flower" concave antiprism, enclosing these annular lateral surfaces by bases which are isotoxal concave *2k*-sided star polygons.
- These star polygons are not stellations (except in two special cases), but elevations of regular *k*-sided convex polygons. This special type of isotoxal star polygon has concave angles that are conjugate to the interior angles of an *n*-sided polygon, where *n=f(k)*, and *n* does not have to be an integer. The number *k* indicates the number of "petals" in the "flower" antiprism itself.
- By an alternative way of folding the net of the lateral surface, we can get another type of "flower" antiprism. If we fold the net so that its central vertices (G_1 and G_2) of the decahedral unit cells are protruding into exterior, we get a lower height of the solid, i.e. the minor type, and if these vertices are indented, we get a greater height of the solid, i.e. the major type. This corresponds to the way all other concave polyhedra of the second sort are formed.
- The bases of the major and minor types will not be congruent in this case, and the aberrations occur in the angles of these polygons, so that the differences will disappear only when $k = \infty$ and *n=5*.
- For the minor type of "flower" antiprisms, conjugate solids are concave antiprisms of the second sort (CA-II-nM), and for the major type, the conjugate solids are double convex antiprisms. For both the major and minor types, there are only 4 cases where both *k* and *n* are of an integer value, whereupon the conjugate solids can modularly fit.

With this research, it is shown that the world of concave polyhedra of the second sort is not depleted with the ones having convex polygons as bases, and that there is a clear relationship between concave antiprisms of the second sort (CA-II-nM), concave “flower” antiprisms (FA-II-k) and convex antiprisms.

For the further research, one might focus on: what happens if we apply the procedure analogous to the one described in this paper to the minor type of the concave antiprisms of the second sort, CA-II-nm, using their fragments for obtaining deltahedral rings? What kind of rings, and “flower” antiprisms would be produced by such an array of their fragments? In this case, even more curiosities occur. Such "flower" antiprisms might even include regular convex polygons as their bases, which is to be explored.

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