Hindawi
Advances in Civil Engineering
Volume 2023, Article ID 5221405, 10 pages
https://doi.org/10.1155/2023/5221405

*Research Article*

# Stability Calculation of the Plane Steel Frame Structures Using Tangent Modulus Theory

**Stanko Ćorić and Zoran Perović**

*Faculty of Civil Engineering, University of Belgrade, Bulevar kralja Aleksandra 73, Belgrade 11000, Serbia*

Correspondence should be addressed to Stanko Ćorić; cstanko@grf.bg.ac.rs

Received 22 March 2023; Revised 7 June 2023; Accepted 27 June 2023; Published 15 July 2023

Academic Editor: Ying Qin



This paper considers the phenomenon of the instability of steel plane frame structures in the elastic–plastic domain. The numerical analysis uses finite element method, where the corresponding stiffness matrices are based upon the trigonometric and hyperbolic interpolation functions of normal forces. The tangent modulus concept is applied when buckling occurs in the plastic range. Thus, the stiffness matrices for nonlinear material behavior are also derived. The procedure for determining effective length factors of compressed columns, which implies a global stability analysis of entire frames, is established too. It means that the proposed approach is based on the assumption that the collapse of the structure occurs when the most loaded columns reach their stability limit. So, the presented procedure enables the calculation of the real critical load of plane frame structures in the elastic–plastic domain and determines the corresponding effective length factors.

## 1. Introduction

The analysis of compressed structural elements requires the investigation of stability phenomena. For example, dominant compressive stresses may cause structural instability, loss of load-bearing capacity, and collapse of the structure, even when allowable stresses are not exceeded. Furthermore, the problems of instability of steel frame structures are current, especially considering the engineers' intentions to design various types of structures. Therefore, the analysis of these structures, particularly taking into account their stability, requires the application of modern and complex numerical methods. That is why, the inelastic buckling of the steel frame columns was always interesting to researchers in structural steel analysis.

The first investigations in this field were based on Euler's theory of buckling of isolated columns [1]. They were mainly based on solving the differential equation of buckling according to the second-order theory. However, in the case of complex structures, it was necessary to introduce some approximations. So, all compressed elements were considered "isolated" from the structure as a whole. These isolated columns are supported only by the adjacent columns and beams. Basically, the corresponding boundary conditions introduce the presence of other structural elements connected to the considered one. Thus, the stability analysis of columns is simplified, and the results may be obtained through the corresponding diagrams and approximate formulas. For example, the results of the elementary case of a column with rotational restraint of stiffness at the ends and restraints of stiffness against sway are presented [2]. The critical loads may be obtained directly from a graph, and the curves are also marked with a ratio of effective length to the actual length. Equivalent diagrams also appear in the European Convention for Constructional Steelwork recommendation [3]. These diagrams became the basis of many design codes, especially for steel structures, for example [4, 5]. Similar approximations also can be applied to the analysis of multistorey frames. The most common procedures for such calculation are the slope deflection method as well as the stiffness distribution method [2, 6]. These hand-calculation methods are easy to apply to simple frames but become difficult, tedious, and time-consuming for frames with many joints. Additionally, some other investigations [7–13] propose methods for analyzing the stability of multistorey multibay frames.

The analysis of "isolated" members was applied to many codes for frame structure stability calculation, such as [4, 5]. The basis of this calculation is the determination of the effective length factor K. Regardless of its frequent use as a basis for design, Chen [14] explained that this approach has significant limitations. The question is whether the coefficient K is determined with adequate accuracy. Namely, Chen [14] indicated no verification of the compatibility between the "isolated" member and the member as part of a whole frame.

That is why, lately, a great effort has been devoted to improving these approximate calculation procedures. So, LeMessurier [15] introduced factors that lead to more realistic results, Gantes and Mageirou [16] proposed improved stiffness distribution factors for effective buckling length calculation, while Tong and Wang [17] considered interstory and intercolumn interactions for the determination of effective length coefficients. In their investigations, Choi et al. [18] used a fictitious axial force factor to determine effective length factors, and Webber et al. [19] and Gunaydin and Aydin [20] improved the calculation of the distribution coefficients.

With advances in computing, the finite element method (FEM) has become one of the most effective methods for analyzing frame stability. This method has been of interest to many researchers, such as Gallagher [21] and Bathe [22], and it is applied in modern commercial software, which deals with stability analysis of frame structures. The common procedure for the FEM analysis is based upon the integral structural model and the geometric stiffness matrix as a part of the tangent stiffness matrix. This study presents an advanced procedure that models the stiffness matrices more accurately by deriving them using the interpolation functions related to the exact solution of the differential equation of the bending of a beam according to the second-order theory. Based on the obtained solutions, the critical load of the frame and the effective length factors are determined by applying a global stability analysis.

Besides the geometrical nonlinearity, this study also considers the physical (or material) nonlinearity. There are various methods to solve this kind of problem. This paper presents tangent modulus theory [23] as one of the most suitable methods for solving such problems. In this method, the tangent modulus, denoted by $E_t$, represents the slope of the stress–strain curve. It is used to describe the stiffness of a material in the plastic range. Its value is a function of the axial load in the observed member, and it can be determined from the capacity specification equations of the column. Therefore, this analysis derives the stiffness matrices of the axially loaded elements using their tangent modulus.

Based on the above description, the most crucial aspect of this paper mainly refers to the investigation of suitable numerical methods for obtaining the solution of the corresponding transcendental stability equation. After finding the appropriate algorithm, the problem was extrapolated to the inelastic material behavior, i.e., to the stability problems in the plastic or the elastic–plastic range. Finally, this study aimed to determine the most appropriate methodology for determining the effective length factors.

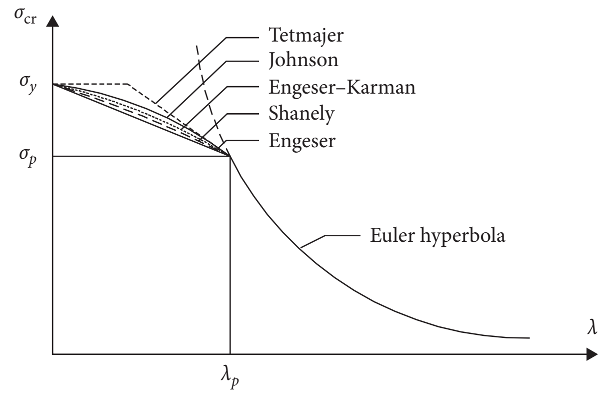


Figure 1: Buckling curves in the plastic domain.

## 2. Buckling Analysis of Plane Frame Structures in the Inelastic Domain

The FEM, as the most convenient for the analysis of the stability of frames, is used in this analysis. In the FEM, the critical load can be obtained as the nontrivial solution of the homogeneous matrix equation:

$$[K] \cdot \{q\} = 0. \tag{1}$$

In Equation (1), $[K]$ represents a global stiffness matrix for the entire frame, including the corresponding boundary conditions, while $\{q\}$ is the vector of generalized coordinates. This matrix equation can be solved incrementally by increasing the load at specified increments until it reaches the critical value, i.e., until $det\ [K] = 0$. When the elastic stability problem is considered, Young's modulus ($E$) has a constant value. Thus, having in mind Euler's formula for the critical load, the member's critical stress can be given as a function of the elasticity modulus and the slenderness ratio:

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 EI}{l_i^2 A} = \pi^2 \frac{E}{\lambda_i^2}, \tag{2}$$

where $\lambda_i = l_i/i$ is the slenderness ratio of the member, $l_i$ is the effective length of the member, $i = \sqrt{I/A}$ is the radius of gyration, $I$ is the moment of inertia, and $A$ is the cross-sectional area. Equation (2) represents a hyperbolic curve, and this function is valid until the critical stress is less than the proportionality limit ($\sigma_p$), as shown in Figure 1. This stability formulation for elastic buckling is given by Allen and Bulson [2] and widely applied in engineering practice. Therefore, regulations for the design of steel structures in the section related to their stability are also based on such an approach [4, 5].

However, this analysis becomes more complex when some compressed columns enter into the phase of nonlinear material behavior, although the critical load still needs to be reached. It means that the stresses in such members exceed the proportional limit value, and buckling occurs in the plastic domain. For that reason, in addition to geometric nonlinearity, material (or physical) nonlinearity should be considered.

This problem has been of interest in many studies. The first test results related to this problem were presented by Bauschinger [24], and von Tetmajer [25] later continued this work and carried out his investigations. As a result, he proposed a linear relationship between critical stress and slenderness ratio in the plastic domain. His solution is still used in engineering practice, and it is shown in Figure 1, where the proportional limit is marked as $\sigma_p$, yield stress with $\sigma_y$, and $\lambda_\mathrm{p} = \pi\sqrt{E/\sigma_p}$ is the slenderness ratio at the proportional limit.

Johnson et al. [26] also proposed a formula that was an alternative to Euler's solution for low slenderness ratio conditions. In his investigations, Engesser [27] involved changing the modulus of elasticity in the plastic domain by introducing a tangent modulus of elasticity. So, he replaced Young's modulus $E$ with the tangent modulus $E_t$ in the expression for Euler's critical load. This problem was also interesting for many other investigations. For example, Von Karman [28] proposed the reduced modulus theory, Shanley [29] claimed that the tangent modulus is correct for buckling beyond the proportional limit, Ramberg and Osgood [30] established an equation that relates Young's modulus to the tangent modulus, etc. Figure 1 also presents some of these significant buckling curves in the plastic range.

Although numerous theoretical and experimental investigations have been carried out, the problem of buckling of frame structures in the elastoplastic domain has yet to be completely solved. However, the rapid development of computer technology in the last decades has enabled the possibility of finding a comprehensive solution to this issue. So, many researchers have continued to deal with this problem, and Chen [14] presented the potential of design procedures in this field. Izzuddin [31] proposed a rotational spring analogy as a simplified method for formulating the geometric stiffness matrix. Furthermore, Yoo and Choi [32] suggested a new method of inelastic buckling analysis, where the concept of modified bifurcation stability using a tangent modulus approach and the column strength curve is applied. Farshi and Kooshesh [33] derived the buckling capacity from a buckling analysis valid for the whole structure and not considered separately and isolated from the rest of the structure.

In order to find an appropriate solution for the buckling problem in the elastoplastic domain, this study applies the tangent modulus theory. Such an approach makes it possible to find the critical load that more closely represents the actual limit state of the structure under pure axial load [23, 34]. So, for each new load increment in all members where the proportional limit is exceeded, the stiffness of the member must be changed. It means that the corresponding tangent modulus $E_t$ has to be used for these members. This modulus represents the slope of the tangent on the stress–strain diagram at any point.

Performing numerical stability analysis in the inelastic domain requires knowledge of the material's mechanical and physical characteristics. It is well known that for building materials such as steel, the $\sigma$–$\varepsilon$ relationship is linear, while it is in the elastic domain of the material. However, this relationship becomes nonlinear with higher load levels, i.e., above the proportional limit. Experimental investigations are the most suitable method of determining this relationship. Figure 2 presents a notional curve that shows the relationship between stress and strain of the axially loaded member before the load-bearing capacity is so reduced that the fracture of structure elements occurs. The notations for $\sigma_p$ and $\sigma_y$ are the same, as shown in Figure 1. Up to a proportional limit $\sigma_p$, Young's modulus of elasticity $E$ is constant and is only a function of the material characteristics. With a further load increase, this modulus also becomes a stress-dependent function called a tangent modulus $E_t$ [35]. It represents the slope of material's stress–strain curve above the proportional limit. Values of $E_t$ can be obtained using either semiempirical formulas derived from inelastic column curves or those derived from the assumed residual stresses in the member [36].

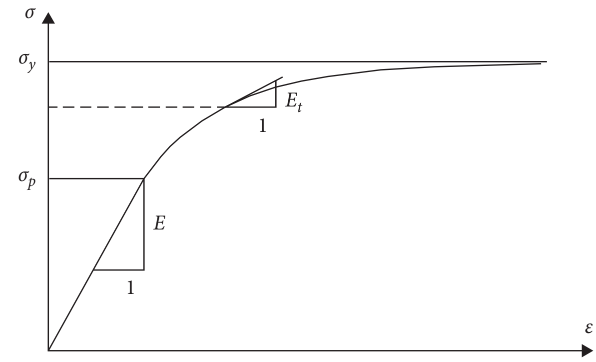

Figure 2: Tangent modulus concept.

Partskhaladze et al. [37] explained that the problem of using an adequate expression for the tangent modulus is relevant. However, this analysis uses an empirical relationship between the two moduli that are suggested in many relevant investigations [34, 38, 39]:

$$E_t = 4E \cdot \left[\frac{\sigma}{\sigma_y}\left(1 - \frac{\sigma}{\sigma_y}\right)\right]. \tag{3}$$

In Equation (3), $\sigma$ is normal stress in axially loaded columns that can be calculated as the quotient of the axial force applied on the column and the cross-sectional area of the column. The material's yield stress $\sigma_y$ is the amount of stress that a material can withstand before undergoing plastic deformation. This empirical formula was derived from inelastic column curves representing the behavior of structural steel columns in the inelastic domain. It was applied in the presented analysis to develop the corresponding computer program that can be used for the nonlinear, i.e., elastoplastic stability analysis of frames.

Since the primary goal of this investigation is the formulation of the exact matrix stability analysis, it is necessary to obtain the corresponding stiffness matrices.

It is well known that during the member buckling, the axial force causes the bending of the member. Thus, the fundamental differential equation of this stability problem in the case when the member is subjected to the axial compressive force $P$ is:

$$v^{iv} + k^2 v'' = 0, \quad (4)$$

where $v$ represents lateral deflection, and $k$ is equal to $\sqrt{P/EI}$ in the case when buckling occurs in the elastic domain.

In the usual approaches based on the FEM, the solutions of the differential Equation (4) can be written as cubic polynomials, suitable for linear static analysis since they are derived from the solution of the differential equation of bending of a beam according to the linear theory. However, the outcome of such investigations showed [40] that significant errors for $P_{cr}$ might be obtained if an insufficient number of finite elements are used. In that case, it is always necessary to control how many finite elements are needed for a convergent solution.

The presented analysis uses interpolation functions in the trigonometric form, related to the exact solution of the basic differential equation of the stability problem. Many other studies dealing with similar topics, such as Rodrigues et al. [41], also apply the same functions. These trigonometric or hyperbolic functions of the axial load in the element are used in the presented form:

$$v(x) = \alpha_1 + \alpha_2 kx + \alpha_3 \sin(kx) + \alpha_4 \cos(kx), \quad (5)$$

where $\alpha_1$, $\alpha_2$, $\alpha_3$, and $\alpha_4$ are integration constants. In the case of the inelastic behavior $k = \sqrt{P/E_t I}$, where $E_t$ is equal to $E$ for the buckling in the elastic domain. Applying such shape functions makes it sufficient to use one finite element for an axially loaded column or beam. However, there are certain disadvantages related to the application of this approach. Namely, instead of the generalized eigenvalue problem, for which there are several well-established methods, the buckling problem is reduced to the solution of the transcendental

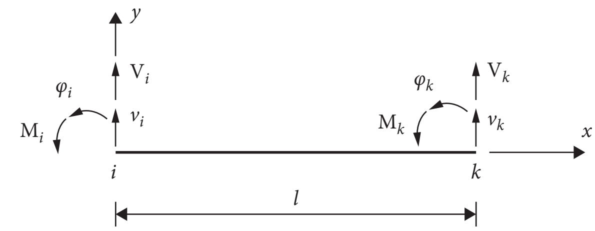


Figure 3: Assumed forces and displacements.

equation that depends, in a highly complex way, upon the axial forces in the columns and beams. So, an essential aspect of this research was devoted to the investigation of suitable numerical methods to obtain the solution of the corresponding transcendental stability equation. These solutions were implemented in a self-developed computer program [40], representing this paper's one of the most important contributions.

The procedure for obtaining the corresponding stiffness matrices for the members subjected to compressive or tension forces is already known, and it will be only briefly presented here. First, it is necessary to determine the values of the integration constants depending on the boundary conditions (see Figure 3).

Then, the displacements within the element are displayed as a function of the generalized displacements at the member ends. This is followed by the calculation of the matrix of interpolation functions. Its elements represent the elastic line of a member clamped on both sides, subjected to the compressive axial force at the ends, due to a unit generalized displacement, while all other displacements are equal to zero. At the end of the procedure, the stiffness matrix $[K]$ makes a correlation between the vector of generalized displacements (i.e., element end shear forces and bending moments) and the vector of generalized displacements (i.e., translational deformations and rotation angles), presented in Figure 3.

So, the matrix for the buckling of the member type "k" (i.e., clamped at both ends), subjected to compressive force in the elastic region, is derived as:

$$[K] = \frac{EI}{l^3 \Delta} \begin{bmatrix} \omega^3 \sin\omega & \omega^2 l(1-\cos\omega) & -\omega^3 \sin\omega & \omega^2 l(1-\cos\omega) \\ & \omega l^2(\sin\omega - \omega\cos\omega) & -\omega^2 l(1-\cos\omega) & \omega l^2(\omega - \sin\omega) \\ & & \omega^3 \sin\omega & -\omega^2 l(1-\cos\omega) \\ & symm. & & \omega l^2(\sin\omega - \omega\cos\omega) \end{bmatrix}, \quad (6)$$

where $\omega = k{\cdot}l$ and $\Delta = 2\,(1-\cos\omega) - \omega\cdot\sin\omega$.

There are many other studies of buckling of Euler–Bernoulli beam columns using this matrix type, for example [42, 43].

This analysis also includes calculating the stiffness matrix when the member buckles in the inelastic domain. In that case, the member stiffness is obtained using the tangent modulus for each member that is consistent with the axial force in the member at the critical load [36]. These matrices have the same form as matrices for linear material behavior but are basically very different. First, it means that a stress-dependent tangent modulus $E_t$ replaces a constant modulus of elasticity $E$. In addition, the values $\omega_t$ and $\Delta_t$ replace $\omega$ and $\Delta$, respectively. So, for the member that is clamped at both ends (so-called type "k") and subjected to compression load, the stiffness matrix can be written as:

$$[K]=\frac{E_tI}{l^3\Delta_t}\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ & C_{22} & C_{23} & C_{24} \\ & & C_{33} & C_{34} \\ & sym. & & C_{44} \end{bmatrix}, \quad (7)$$

where is:

$$E_t=4E\cdot\left[\frac{\sigma}{\sigma_y}\cdot\left(1-\frac{\sigma}{\sigma_y}\right)\right]=4E\cdot\left[\frac{P_i}{A\cdot\sigma_y}\cdot\left(1-\frac{P_i}{A\cdot\sigma_y}\right)\right], \quad (8)$$

$$\Delta_t=2\cdot(1-\cos\omega_t)-\omega_t\cdot\sin\omega_t, \quad (9)$$

$$\omega_t=k\cdot l=\sqrt{\frac{P_i}{E_t\cdot I}}\cdot l=\frac{1}{2}A\,\sigma_y\,l\cdot\sqrt{\frac{1}{EI\left(A\sigma_y-P_i\right)}}, \quad (10)$$

$$C_{11}=C_{33}=-C_{13}=\omega_t^3\sin\omega_t, \quad (11)$$

$$C_{12}=C_{14}=-C_{23}=-C_{34}=\omega_t^2l(1-\cos\omega_t), \quad (12)$$

$$C_{22}=C_{44}=\omega_t\,l^2(\sin\omega_t-\omega_t\cos\omega_t), \quad (13)$$

$$C_{24}=\omega_t\,l^2(\omega_t-\sin\omega_t). \quad (14)$$

In these equations, $P_i$ is an axial force for the observed element. Column length is denoted by $l$, $I$ is the moment of inertia, $A$ is the cross-sectional area, and $\sigma_y$ is the yield stress. The stiffness matrix of the member that is clamped at one and hinged at the other end (so-called type "g") is given by Ćorić [40]. When the element is subjected to a tensile load, stiffness matrices have a similar form; only the hyperbolic functions are used instead of the trigonometric one.

As was already emphasized, the calculation methodology used in this analysis relates to the global stability analysis of plane frame structures. It means that first, it is necessary to determine the critical buckling load for the whole structure ($P_{cr,gl}$). Then, it is possible to find the critical force for each column of the analyzed frame ($P_{cr,i}$) when the whole frame buckles.

It should be noted that this investigation assumes that all members reach their buckling limits when the buckling of the overall structure occurs. So, when the critical load of the whole structure is obtained, the buckling load of each member in the frame can be found as follows:

$$P_{cr,i}=n\cdot P_{cr,gl}, \quad (15)$$

where $n$ denotes the ratio of the axial force in the column in relation to the load parameter $P$. For example, if the axial force in the column is $2P$, then $n=2$, etc.

Based on the presented approach, for the elastic stability analysis, the effective length coefficient of the individual column is given by:

$$K=\sqrt{\frac{\pi^2EI}{P_{cr}\cdot l^2}}. \quad (16)$$

When the buckling occurs in the plastic range, this expression for $K$ has the same form, as shown in Equation (16), but $P_{cr}$ refers to the critical load obtained in the inelastic stability analysis, denoted as $P_{cr,inel}$. In the same way, the tangent modulus $E_t$ replaces the elastic modulus $E$:

$$K=\sqrt{\frac{\pi^2E_tI}{P_{cr,inel}\cdot l^2}}. \quad (17)$$

A computer program in the C++ programming language was developed to implement the presented theoretical approach. Two steps perform the proposed analysis. The first step requires stability calculation in the elastic domain. The critical buckling load is obtained from the roots of the transcendental equation, representing the condition that the determinant of the corresponding stiffness matrix is equal to zero. This calculation is performed iteratively. Namely, the initial values of the load factor are set, and the calculation is repeated until a preset accuracy is reached. Finally, the critical force in the elastic domain is obtained. Then, the critical stress for all members is calculated as the ratio of the critical normal force and the cross-sectional area of the analyzed element, as shown in Equation (2). For the axially loaded columns in which the critical stress exceeds the proportionality limit ($\sigma_p$), it was necessary to change their stiffness. So, a new tangent modulus ($E_t$) was taken in the form (3). On the other hand, columns with critical stress, which did not reach the proportionality limit, keep their "old characteristics." So, the stiffness matrix for this element is the same as in the first part of this calculation.

Additionally, possible geometric imperfections can be simulated by a further reduction of member stiffness. As suggested by Chen [14], it can be achieved by an additional reduction of the tangent modulus ($E_t'=0.85E_t$). The calculation is again performed iteratively, and the obtained reduced stiffness matrix must satisfy the condition for the existence of the nontrivial solution. As a result of this procedure, the corresponding critical load factor, effective length coefficient, and the value of tangent modulus for all elements buckling in the inelastic domain are obtained.

## 3. Behavior of Frame Structures in the Elastoplastic Domain

Due to its complexity, the stability calculation of the frame structure in the elastoplastic domain is not represented in standard engineering calculations. Therefore, this kind of calculation is not required by the current codes for the design of steel structures. Namely, to obtain the critical load and

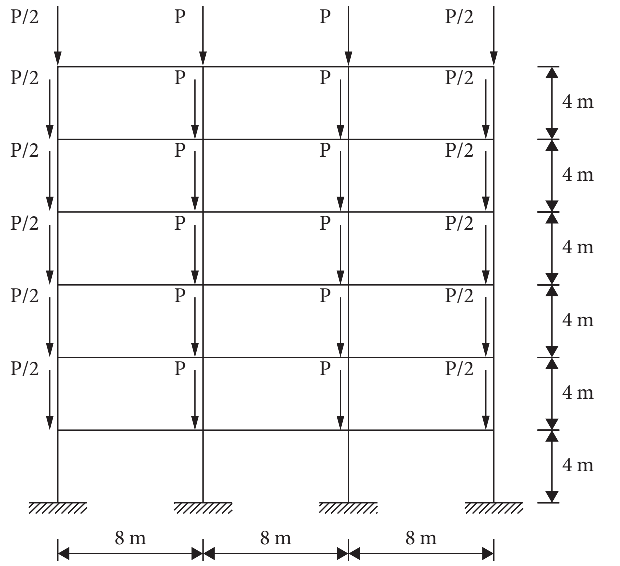


FIGURE 4: Geometry and the loading data for the six-story three-bay plane sway frame.

stresses in the plastic range, the regulations suggest that calculation in the elastic domain should be carried out first. Then, it is necessary to apply the corresponding buckling expressions and curves that arise from numerous experimental investigations related to the behavior of the isolated members in the plastic domain. This study will show that the stability analysis of the frame structures in the elastoplastic domain could be performed more correctly using the presented theoretical approach.

In order to illustrate the proposed method, a three-bay plane frame of six stories is considered. Figure 4 shows that this sway frame is clamped at the base, and external loads $P$, i.e., $P/2$, are applied to each column at each story in the frame. In this numerical example, the axial load in the columns increases from the upper stories to the primary level. It means that the elastoplastic analysis of stability can result in the different behavior of the columns in different stories.

The considered frame is made of steel with a modulus of elasticity $E = 210{,}000{,}000\,\mathrm{kN/m^2}$ and yield stress $\sigma_y = 240{,}000\,\mathrm{kN/m^2}$. From the tangent modulus formula, Equation (3), it follows that the proportional limit is half the yield point, i.e., $\sigma p = 0.5{\cdot}\sigma y = 120{,}000\,\mathrm{kN/m^2}$ [34, 38, 39]. In other words, this tangent modulus formula adjustment affects members with $\sigma/\sigma_y > 0.5$, as indicated by Ziemain [36]. Several different types of sections are assumed for the columns in this numerical example (IPB140, IPB180, IPB220, IPB260, IPB300, IPB340), while IPE360 is used for all girders.

The stability analysis results for this frame are given in Tables 1 and 2. First, Table 1 shows the critical load values for all considered cross-sections in the case of elastic and proposed elastoplastic stability analysis.

It is clear that elastoplastic stability analysis, compared to the classical elastic analysis, gives smaller values of the critical force, as expected. Moreover, this difference in the results rises with the increase of the stiffness of the axially loaded columns.

TABLE 1: Critical load values for the six-story three-bay sway frame (kN).

| | Elastic analysis | Elastoplastic analysis |
|---|---|---|
| IPB140 | $P_{cr,el} = 405.59$ | $P_{cr,inel} = 168.58$ |
| IPB180 | $P_{cr,el} = 905.27$ | $P_{cr,inel} = 255.98$ |
| IPB220 | $P_{cr,el} = 1{,}524.52$ | $P_{cr,inel} = 359.19$ |
| IPB260 | $P_{cr,el} = 2{,}119.72$ | $P_{cr,inel} = 467.61$ |
| IPB300 | $P_{cr,el} = 2{,}652.27$ | $P_{cr,inel} = 591.86$ |
| IPB340 | $P_{cr,el} = 3{,}049.66$ | $P_{cr,inel} = 682.16$ |

TABLE 2: Comparison of the tangent modulus for the six-story three-bay sway frame $E_t$ (kN/m$^2$).

| | First floor | Second floor | Third floor |
|---|---|---|---|
| IPB140 | 18,375,849 | 126,718,202 | 190,251,891 |
| IPB180 | 16,437,448 | 125,752,069 | 190,245,012 |
| IPB220 | 10,943,660 | 122,726,142 | 189,066,054 |
| IPB260 | 7,732,187 | 120,952,255 | 188,368,916 |
| IPB300 | 5,801,178 | 119,883,901 | 187,946,789 |
| IPB340 | 2,257,235 | 117,919,839 | 187,166,919 |

Based on these results, it is obvious that only inelastic stability analysis is acceptable for the analyzed example. In that case, the physical characteristics of the steel are changed, and the corresponding tangent modules $E_t$ should be considered. Table 2 summarizes the results for these two modules when buckling occurs. As the columns are stiffer, they can receive a greater load, so their tangent modulus values are lower. So, Table 2 contains the results only for the three most loaded stories of the analyzed frame structure. The results are given for the inner columns, considering they are loaded with the double axial forces loading than the outer ones within one floor.

## 4. Determination of Effective Length Factors of Compression Members

Investigating the collapse of some buildings, particularly steel ones, has shown that it often occurs due to the fracture of their compression elements. These elements suffered a premature fracture, so their load-bearing capacity was exhausted even before the allowable stresses were reached. Generally, for most of the compression members made of steel, the ultimate bearing capacity is determined by the stability criterion.

Many standards for the calculation of centrally loaded steel members, as well as European standards EC3 [5], are based on the application of the so-called "buckling curves." These curves are obtained from the theoretical buckling curves considering the geometric imperfections and residual stresses in real members. They are represented in the buckling analysis by a reduction factor and the slenderness ratio [5]. So, to use these curves properly and get the correct results, it is necessary to calculate the analyzed members'

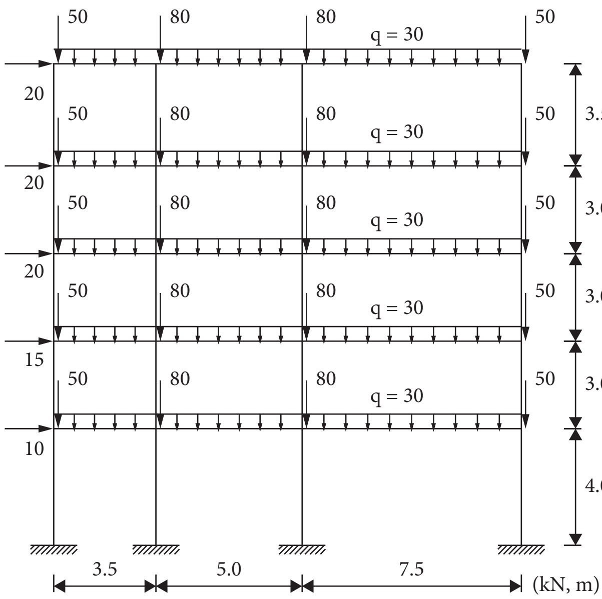


FIGURE 5: Numerical example—five-story three-bay sway frame [33].

buckling length precisely. It means that the accurate value of the effective length factor ($K$) can be a critical parameter for the design of steel frame structures.

From the physical viewpoint, this "effective buckling length" is the length of an equivalent end-hinged member with the same critical force as the member under consideration. The critical force is given by:

$$P_{cr} = \frac{\pi^2 EI}{(K \cdot L)^2}, \tag{18}$$

where the effective length of the column is determined as the multiplication of the effective length factor ($K$) and the unsupported length of the column ($L$). Mathematically speaking, "effective buckling length" is the distance between the inflection points of the member subjected to the compression force. This length depends on the support conditions at the end of the column.

The effective length factor was determined using the theoretical approach described in Section 2 of this paper instead of the approximate procedure from the codes. Namely, structural design codes for the stability calculation of steel structures use simplified static schemes to analyze the compressed column within a frame. It means that only "isolated" columns supported by the adjacent beams and columns are considered. In essence, the corresponding boundary conditions introduce the presence of other structural elements connected to the considered one. In many studies [16, 19, 40], however, it has been shown that these simplified solutions are not accurate enough to calculate columns within frames.

The selected test model for calculating the effective length factors for axially loaded columns is presented in Figure 5, and it has been considered and investigated by Farshi and Kooshesh [33].

It is a five-story three-bay sway frame that is clamped at the base. The used material is steel with properties: $E = 210{,}000{,}000\,\text{kN/m}^2$ and $\sigma_y = 240{,}000\,\text{kN/m}^2$. The frame dimensions and the applied loads are shown in Figure 5, with length in meters (m) and loads in kN. The left-side and right-side columns on the three higher floors are IPB300, while the first two are IPB340. The middle columns on the three higher floors are IPB340, while the middle on the first two floors are IPB400. The girders of the left, middle, and right bays are 2IPE240, 2IPE270, and 2IPE300, respectively. The columns are numbered from the lower to the upper floors (1–4 first level and, respectively to, 17–20 on the highest level), and within each floor from left to right (e.g., on the first floor, 1 is the left column, 2 and 3 are the inner ones, and 4 is the outer right column).

As in the previous example, the initial analysis assumes that all members are elastic, i.e., have a constant modulus of elasticity $E$. Considering that some columns have the critical stress that exceeds the proportionality limit ($\sigma_p$), it was necessary to perform an inelastic analysis. The calculation was carried out according to the procedure presented in Section 2. The buckling load factor and the tangent modulus were calculated for all columns that behave inelastically. As expected, the less axially loaded columns continue to behave elastically. These are those columns on the upper floors and the outer ones on the left side of the frame. Also, girders with low axial stress would continue to behave elastically. However, more loaded columns, i.e., those on the lower levels, primarily the inner ones, enter into the plastic range. So, they changed their characteristics and the corresponding tangent modulus $E_t$ was calculated.

The effective buckling length factors were calculated using Equations (16) and (17). These values are compared with the results from Farshi and Kooshesh [33] and with the results from regulation EC3 [5], and they are presented in Table 3. Farshi and Kooshesh's [33] procedure also considered that many columns show inelastic behavior at the buckling stage. Also, design parameters related to the buckling capacity are derived from the buckling analysis that is valid for the whole structure and not considered separately and isolated from the rest of the structure. Table 3 also gives differences between the proposed and other two approaches (in %). These differences are calculated according to the formula for relative error, as done in [44]: $(K^{\#}-K)/K\cdot 100\%$, where $K$ is the result of the presented analysis and $K^{\#}$ is the result of the Farshi procedure and EC3 code.

First, there is quite a good correlation between the proposed and Farshi study results. Namely, both methods have a similar approach based on the global stability analysis using the FEM. Interestingly, the best matching results with these two methods are for the first (most loaded) floor. Specific differences in the effective length factors are observed for columns with higher floors. The authors believe that this is because the expression for the tangent modulus used in this analysis, as shown in Equation (3), better describes the material's behavior than the reduced modulus used by Farshi and Kooshesh [33].

Moreover, it is clear that there is a significant difference between these results and the results obtained using the codes

Table 3: Comparison of the effective length factors $K$ for all columns of the numerical example (Figure 5) obtained by different methods.

| Column number | Presented analysis | Farshi [33] (difference) | EC3 [5] (difference) |
|---|---|---|---|
| 1 | 2.20 | 2.01 (−9%) | 1.42 (−36%) |
| 2 | 1.46 | 1.37 (−6%) | 1.29 (−12%) |
| 3 | 1.25 | 1.19 (−5%) | 1.29 (3%) |
| 4 | 1.58 | 1.46 (−8%) | 1.44 (−9%) |
| 5 | 3.52 | 2.95 (−16%) | 2.04 (−42%) |
| 6 | 2.45 | 2.05 (−16%) | 1.65 (−33%) |
| 7 | 2.11 | 1.77 (−16%) | 1.65 (−22%) |
| 8 | 2.48 | 2.18 (−12%) | 2.08 (−16%) |
| 9 | 4.27 | 3.16 (−26%) | 2.05 (−52%) |
| 10 | 3.04 | 2.25 (−26%) | 1.64 (−46%) |
| 11 | 2.58 | 1.93 (−25%) | 1.66 (−36%) |
| 12 | 3.18 | 2.38 (−25%) | 2.05 (−36%) |
| 13 | 5.44 | 3.82 (−30%) | 2.00 (−63%) |
| 14 | 3.93 | 2.76 (−30%) | 1.63 (−59%) |
| 15 | 3.35 | 2.36 (−30%) | 1.63 (−51%) |
| 16 | 4.14 | 2.93 (−29%) | 1.99 (−52%) |
| 17 | 5.66 | 4.61 (−19%) | 1.73 (−69%) |
| 18 | 4.15 | 3.36 (−19%) | 1.47 (−65%) |
| 19 | 3.48 | 2.84 (−18%) | 1.48 (−57%) |
| 20 | 4.38 | 3.59 (−18%) | 1.73 (−61%) |

for steel structures. Likewise, the AISC procedure [45] gives similar results to the Eurocode. It is understandable because those procedures, according to the regulations, are based on many simplifications. First, as already emphasized, they only consider the stiffness of the structural elements connected to the analyzed one. In addition, codes do not consider the value of the axial force in the observed element. Also, EC3 proposes only the elastic stability analysis for determining effective length factors.

## 5. Conclusions

The presented procedure suggests a global stability analysis of the whole steel frame structure. First, it is necessary to determine the critical load for the structure as a whole. Then, each member's critical force and effective length factors can be obtained based on the relationship between the global critical force and the axial forces in the individual elements. Matrix analysis applied to this analysis is based on applying trigonometric shape functions. So, the presented method is not unknown, but the way it is formulated and implemented here has not been used so far in any software dealing with the stability calculation of frame structures.

The previously mentioned approach has also been used for inelastic analysis, where the nonlinear material behavior is also developed apart from geometric nonlinearity. In this case, the corresponding stiffness matrices have been derived using the tangent modulus theory. These matrices were implemented in the self-developed computer code. An example of the six-story three-bay frame was used to present the advantages of such analysis.

This analysis also investigated the methodology for determining the effective length factors. Namely, the concept of an effective length is most commonly used to calculate axially loaded columns. So, it is necessary to accurately calculate effective length factors, both in the elastic and inelastic domains. A five-story three-bay frame already used in other investigations was chosen as a benchmark numerical example. Based on the obtained results, the validity of the recommended inelastic buckling analysis can be verified.

The results obtained in this analysis confirm that the proposed inelastic buckling approach is convenient for determining the critical load and the effective buckling length of the steel frame structures. Therefore, it can be used as a good alternative for estimating the load-bearing capacity of axially loaded elements in the design of steel frames.

Finally, it should be emphasized again that the primary goal of this research was to formulate suitable methods to obtain more efficient and reliable solutions to stability problems to frame structures in the inelastic domain. So, its main contribution is related to the calculation algorithm that is implemented in the self-developed code. This algorithm results in more precise buckling calculations both in the elastic and the inelastic domains. Furthermore, it enables monitoring the loss of structural stability in the plastic range and determining the critical load when the frame structure buckles.

## Data Availability

The data used to support the findings of this analysis are available from the corresponding author upon request.

## Disclosure

The part of this investigation was presented at the 7th International Congress of the Serbian Society of Mechanics [46].

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

## Authors' Contributions

Stanko Coric contributed to the conceptualization, methodology, investigation, formal analysis, numerical analysis, original draft preparation, and visualization of this manuscript. Zoran Perovic contributed to the data curation, results analysis, and editing of this manuscript. Both authors read and approved the final manuscript.

## Acknowledgments

This research was supported by the Ministry of Education, Science, and Technological Development, Republic of Serbia, through Project 200092.

## References

[1] S. P. Timoshenko and J. M. Gere, *Theory of Elastic Stability*, McGraw-Hill, New York, 1965.

[2] H. G. Allen and P. S. Bulson, *Backgroung to Buckling*, McGraw Hill, U.K., 1980.

[3] ECCS, *European Recommendations for Steel Construction: The Stressed Skin Design of Steel Structures*, ECCS, 1978.

[4] European Committee for Standardization, *Eurocode 3, Design of Steel Structures—ENV 1993-1-1*, European Commission, 1992.

[5] European Committee for Standardization, *Eurocode 3, Design of Steel Structures—ENV 1993-1-1*, European Commission, 2006.

[6] M. R. Horne and W. Merchant, *The Stability of Frames*, Pergamon Press, Oxford, 1965.

[7] R. H. Wood, "Effective lengths of columns in multi-storey buildings," *The Structural Engineer*, vol. 52, no. 7, pp. 235–244, 1974.

[8] R. H. Wood, "Effective lengths of columns in multi-storey buildings," *The Structural Engineer*, vol. 52, no. 8, pp. 295–302.

[9] R. H. Wood, "Effective lengths of columns in multi-storey buildings," *The Structural Engineer*, vol. 52, no. 9, pp. 341–346, 1974.

[10] R. H. Wood, "Historical paper: the stability of tall buildings," *Proceedings of the Institution of Civil Engineers—Structures and Buildings*, vol. 161, no. 5, pp. 247–263, 2008.

[11] M. R. Horne, "An approximate method for calculating the elastic critical loads of multi-storey plane frames," *The Structural Engineer*, vol. 53, no. 6, pp. 242–248, 1975.

[12] D. W. White and J. F. Hajjar, "Buckling models and stability design of steel frames: a unified approach," *Journal of Constructional Steel Research*, vol. 42, no. 3, pp. 171–207, 1997.

[13] D. W. White and J. F. Hajjar, "Accuracy and simplicity of alternative procedures for stability design of steel frames," *Journal of Constructional Steel Research*, vol. 42, no. 3, pp. 209–261, 1997.

[14] W. F. Chen, "Structural stability: from theory to practice," *Engineering Structures*, vol. 22, no. 2, pp. 116–122, 2000.

[15] W. J. LeMessurier, "A practical method of second-order analysis," *Journal AISC*, vol. 14, no. 2, pp. 48–67, 1972.

[16] C. J. Gantes and G. E. Mageirou, "Improved stiffness distribution factors for evaluation of effective buckling lengths in multi-story sway frames," *Engineering Structures*, vol. 27, no. 7, pp. 1113–1124, 2005.

[17] G. S. Tong and J. P. Wang, "Column effective lengths considering inter-story and inter-column interactions in sway-permitted frames," *Journal of Constructional Steel Research*, vol. 62, no. 5, pp. 413–423, 2006.

[18] D.-H. Choi, H.-S. Na, J.-W. Jung, and H. Yoo, "Iterative eigenvalue analysis for stability design of three-dimensional frames considering a fictitious axial force factor," *International Journal of Steel Structures*, vol. 10, pp. 325–335, 2010.

[19] A. Webber, J. J. Orr, P. Shepherd, and K. Crothers, "The effective length of columns in multi-storey frames," *Engineering Structures*, vol. 102, pp. 132–143, 2015.

[20] A. Gunaydin and R. Aydin, "A simplified method for instability and second-order load effects of framed structures: story-based approach," *The Structural Design of Tall and Special Buildings*, vol. 28, no. 13, Article ID e1655, 2019.

[21] R. H. Gallagher, *Finite Element Analysis*, Prentice-Hall, Inc., New Jersey, 1975.

[22] K.-J. Bathe, *Finite Element Procedures in Engineering Analysis*, Prentice-Hall, 1982.

[23] T. V. Galambos, *Guide to Stability Design Criteria for Metal Structures*, John Wiley and Sons, New York, USA, 5th edition, 1985.

[24] J. Bauschinger, "Über die veränderung der elastizitätsgrenze und des elastizitätsmoduls verschiedener metalle," *Civilingenieur*, vol. 27, pp. 289–348, 1881.

[25] L. von Tetmajer, *Die Gesetze der Knickung- und der Zusammengesetzten Druckfestigkeit der Technisch Wichtigsten*, Franz Deuticke, Leipzig und Wien, 3rd edition, 1903.

[26] J. B. Johnson, C. W. Brayan, and F. E. Turneaure, *The Theory and Practice of Modern Framed Structures*, Wiley, New York, 1894.

[27] F. Engesser, "Ueber die Knickfestigkeit Gerader Stabe," *Zeistschrift fur Architektur und Ingenieurwesen*, vol. 35, pp. 455–462, 1889.

[28] T. Von Karman, "Untersuchungen uber die Knickfestigkeit," in *Mitteilungen uber Forschungsarbeiten auf dem Gebiete des Ingenieurwensen*, Springer, 1910.

[29] R. F. Shanley, "Inelastic column theory," *Journal of the Aeronautical Sciences*, vol. 14, no. 5, pp. 261–268, 1947.

[30] W. Ramberg and W. R. Osgood, "Description of stress–strain curves by three parameters," Technical Note No. 902, National Advisory Committee for Aeronautics, Washington DC, 1943.

[31] B. A. Izzuddin, "Simplified buckling analysis of skeletal structures," *Proceedings of the Institution of Civil Engineers—Structures and Buildings*, vol. 159, no. 4, pp. 217–228, 2006.

[32] H. Yoo and D.-H. Choi, "New method of inelastic buckling analysis for steel frames," *Journal of Constructional Steel Research*, vol. 64, no. 7, pp. 1152–1164, 2008.

[33] B. Farshi and F. Kooshesh, "Buckling analysis of structural steel frames with inelastic effects according to codes," *Journal of Constructional Steel Research*, vol. 65, no. 10-11, pp. 2078–2085, 2009.

[34] W. McGuire, R. H. Gallagher, and R. D. Ziemain, *Matrix Structural Analysis*, John Wiley and Sons, New York, USA, 2nd edition, 2000.

[35] Z. P. Bažant and L. Cedolin, *Stability of Structures: Elastic, Inelastic, Fracture, and Damage Theories*, Dover Publications, Inc., Mineola, New York, 2003.

[36] R. D. Ziemain, *Guide to Stability Design Criteria for Metal Structures*, John Wiley & Sons, New Jersey, USA, 2010.

[37] G. Partskhaladze, I. Mshvenieradze, E. Medzmariashvili, G. Chavleshvili, V. Yepes, and J. Alcala, "Buckling analysis and stability of compressed low-carbon steel rods in the elastoplastic region of materials," *Advances in Civil Enginerring*, vol. 2019, Article ID 7601260, 9 pages, 2019.

[38] Column Research Committee of Japan, *Handbook of Structural Stability*, Corona, Tokyo, 1971.

[39] R. D. Ziemain and W. McGuire, "Modified tangent modulus approach, a contribution to plastic hinge analysis," *Journal of Structural Engineering*, vol. 128, no. 10, pp. 1301–1307, 2002.

[40] S. Ćorić, "*Nonlinear stability analysis of the frame structures*," Ph.D. dissertation, Faculty of Civil Engineering University of Belgrade, 2013.

[41] M. A. C. Rodrigues, R. B. Burgos, and L. F. Martha, "A unified approach to the Timoshenko 3D beam-column element tangent stiffness matrix considering higher-order terms in the strain tensor and large rotations," *International Journal of Solids and Structures*, vol. 222-223, Article ID 111003, 2021.

[42] Z.-P. Hu, W.-H. Pan, and J.-Z. Tong, "Exact solutions for buckling and second-order effect of shear deformable timoshenko beam–columns based on matrix structural analysis," *Applied Sciences*, vol. 9, no. 18, Article ID 3814, 2019.

[43] S. Ćorić and S. Brčić, "Column buckling investigation of plane frames," in *Proceedings of the 2nd International Congress of Serbian Society of Mechanics C-08:1-12*, Serbian Society of Mechanics, Palić, Serbia, June 2009.

[44] X. Zhai, X. Zha, and D. Chen, "Elastic stability of unbraced plate-type modular steel frames with semi-rigid corner connections," *Journal of Constructional Steel Research*, vol. 192, Article ID 107243, 2022.

[45] American Institute of Steel Construction, *Specification for Structural Steel Buildings*, American Institute of Steel Construction, 2016.

[46] S. Ćorić, "Stability analysis of multy-story steel frames subjected to different axial load," in *Proceedings of the 7th International Congress of Serbian Society of Mechanics*, Serbian Society of Mechanics, Sremski Karlovci, Serbia, June 2019.