Discrete Mathematics 313 (2013) 2192–2194

Contents lists available at SciVerse ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Note

The maximum number of P-vertices of some nonsingular double star matrices

Aleksandra Erić [a], C.M. da Fonseca [b,*]

[a] *Faculty of Civil Engineering, University of Belgrade, 11000 Belgrade, Serbia*

[b] *Departamento de Matemática, Universidade de Coimbra, 3001-501 Coimbra, Portugal*

ARTICLE INFO

Article history:
Received 22 February 2012
Received in revised form 21 May 2013
Accepted 22 May 2013
Available online 15 June 2013

Keywords:
Graphs
Matrices
Eigenvalues
Double stars
P-vertices

ABSTRACT

In this short note, we construct a nonsingular matrix A whose graph is a double star of order $n \geqslant 4$ with $n-2$ P-vertices. This example leads to a positive answer, for $n \geqslant 6$, to a last open question proposed recently by Kim and Shader regarding the trees for which each nonsingular matrix has at most $n-2$ P-vertices.

1. Introduction

For a given $n \times n$ real symmetric matrix $A = (a_{ij})$, we define the graph of A, which we write as $G(A)$, as the (simple) graph whose vertex set is $\{1, \ldots, n\}$ and edge set is $\{ij \mid i \neq j \text{ and } a_{ij} \neq 0\}$. We confine our attention to the set

$$\mathcal{S}(G) = \{A \in \mathbb{R}^{n\times n} \mid A \text{ is symmetric and } G(A) = G\},$$

i.e., the set of all symmetric matrices sharing a common graph G on n vertices. If G is a tree, then $A \in \mathcal{S}(G)$ is an irreducible *acyclic* matrix.

Let us denote the (algebraic) multiplicity of the eigenvalue θ of a symmetric matrix A by $m_A(\theta)$. By $A(i)$ we mean the $(n-1) \times (n-1)$ principal submatrix formed by the deletion of the row and column indexed with i. More generally, if S is a subset of the vertex set of G, then $A(S)$ is the principal submatrix obtained from A by striking out rows and columns S. By $A[S]$ we mean the principal submatrix of A whose rows and columns are indexed with S. The reader is referred to [8–10] for a full account regarding the terminology used throughout.

Probably the main consequence of Cauchy's Interlacing Theorem for the eigenvalues of symmetric matrices is the set of inequalities

$$m_A(\theta) - 1 \leqslant m_{A(i)}(\theta) \leqslant m_A(\theta) + 1.$$

In the case of $m_{A(i)}(\theta) = m_A(\theta) + 1$, the vertex i is known as a *Parter-vertex* of A for θ [8–10] or as a θ-*positive* vertex of G [3,5,6]. When $\theta = 0$, a Parter-vertex is simply called a *P-vertex* of A [9], and $P_\nu(A)$ denotes the number of P-vertices of A.

In 2004, Johnson and Sutton [7] showed that each singular acyclic matrix of order n has at most $n-2$ P-vertices. Later, Kim and Shader proved in [8] that this does not hold for nonsingular acyclic matrices by constructing some examples for

* Corresponding author.
E-mail addresses: eric@grf.rs (A. Erić), cmf@mat.uc.pt (C.M. da Fonseca).

0012-365X/$ – see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.disc.2013.05.018

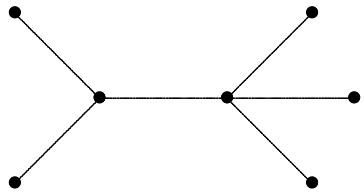

Fig. 1. The double star S_{34}.

paths and stars. Furthermore, these authors proved that $P_\nu(A) \leqslant n-1$, for any nonsingular matrix A in $\mathcal{S}(T)$, when n is odd. Clearly, when n is even, we have $P_\nu(A) \leqslant n$. More recently, Anđelić et al. [2,1] considered other general cases and discussed some "continuity" properties of $P_\nu(A)$, when A runs over all tridiagonal matrices, for example.

One of the questions left open by Kim and Shader [8, Question (g), p. 407] concerned the existence of a tree T, of order n, such that for each nonsingular matrix $A \in \mathcal{S}(T)$, $P_\nu(A) \leqslant n-2$. In this brief note, we provide a positive answer to this question. More precisely, using an elementary approach, we show that a double star of order $\geqslant 6$ satisfies such an inequality.

2. Double stars

Let us recall that a *double star* is the tree obtained from two vertex disjoint stars by connecting their centers by a path. Double stars emerge often in the literature and constitute an important family of acyclic graphs [4,10]. Here we will consider two stars whose central vertices are joined by an edge. In order to be more precise, we write $S_{k_1k_2}$, with $k_1+k_2=n$, specifying the sizes of the two "disjoint" stars (see Fig. 1). In particular, a star on n vertices is a double star of the form $S_{n-1,1}$.

Let us consider now for $n \geqslant 4$ the matrix

$$A_n = \left(\begin{array}{cccc|ccc} 1 & & & 1 & & & \\ & \ddots & & \vdots & & & \\ & & 1 & 1 & & & \\ 1 & \cdots & 1 & n-5 & 1 & & \\ \hline & & & 1 & 0 & 1 & 1 \\ & & & & 1 & 1 & 0 \\ & & & & 1 & 0 & 0 \end{array}\right),$$

where the upper left block is of order $n-3$.

We first observe that $\det A_n = 1$. On the other hand,

$$\det A_n(\ell) = 0, \quad \text{for } \ell \in \{1, \dots, n\} - \{n-3, n-1\}$$

and

$$\det A_n(n-1) = -\det A_n(n-3) = 1,$$

i.e.,

$$P_\nu(A_n) = n-2.$$

We point out that the graph of A_n is the double star $S_{n-3,3}$.

3. The main result

We start this main section by observing that, for any $n = 2, 3, 4, 5$, it is possible to construct a nonsingular acyclic matrix whose graph is a path or a star of order n, with $n-1$ or n P-vertices (see [1]). Moreover, for $n = 5$, the nonsingular matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

has four P-vertices. In this case, the graph of A is the double star S_{23}. Therefore, for $n \leqslant 5$, the answer to Question (g) posed by Kim and Shader [8] is "no". But, for $n \geqslant 6$, our result provides a positive answer to that question.

Theorem 3.1. *For any nonsingular matrix $A \in \mathcal{S}(S_{n-3,3})$, with $n \geqslant 6$,*

$$P_\nu(A) \leqslant n-2.$$

Proof. We begin by noting that, for any nonsingular matrix $A = (a_{ij}) \in \mathcal{S}(S_{n-3,3})$, $P_\nu(A) \leqslant n-1$ [1, Theorem 7.1]. Furthermore, at least one of the vertices $n-2$, $n-1$, n is not a P-vertex. In fact, let us assume that those vertices are all P-vertices. Consequently

$$\det A = -a_{n,n-2}^2 a_{n-1,n-1} \det A(n-2, n-1, n) \neq 0, \tag{3.1}$$

$$\det A = -a_{n,n-1}^2 a_{n,n} \det A(n-2, n-1, n) \neq 0, \tag{3.2}$$

and

$$\det A(n-2) = a_{n-1,n-1} a_{n,n} \det A(n-2, n-1, n) = 0.$$

So, if $a_{nn} \neq 0$, we get a contradiction with (3.1); otherwise, $a_{nn} = 0$ will contradict (3.2).

Now, let us suppose that $P_\nu(A) = n-1$. Since the vertex $n-3$ is one of the centers of the double star, we have

$$0 = \det A(n-3) = a_{11} a_{22} \cdots a_{n-4,n-4} \det A[n-2, n-1, n]$$

and, on the other hand,

$$0 \neq \det A = -a_{1,n-3}^2 a_{22} \cdots a_{n-4,n-4} \det A[n-2, n-1, n],$$

because $\det A(1) = 0$. Therefore

$$a_{11} = 0.$$

But, since $\det A(2) = 0$, we also have

$$\det A = -a_{2,n-3}^2 a_{11} a_{33} \cdots a_{n-4,n-4} \det A[n-2, n-1, n] = 0,$$

which contradicts the nonsingularity of A. Taking into account the discussion in the previous section, the result follows. □

Acknowledgments

We thank both referees for providing valuable and constructive comments which helped us to improve and correct the original manuscript.

References

[1] M. Anđelić, C.M. da Fonseca, R. Mamede, On the number of P-vertices of some graphs, Linear Algebra Appl. 434 (2) (2011) 514–525.
[2] M. Anđelić, A. Erić, C.M. da Fonseca, Nonsingular acyclic matrices with full number of P-vertices, Linear Multilinear Algebra 61 (1) (2013) 49–57.
[3] C.M. da Fonseca, On the multiplicities of eigenvalues of a Hermitian matrix whose graph is a tree, Ann. Mat. Pura Appl. 187 (2) (2008) 251–261.
[4] A. Erić, C.M. da Fonseca, Some consequences of an inequality on the spectral multiplicity of graphs, Filomat (in press).
[5] C.D. Godsil, Spectra of trees, Ann. Discrete Math. 20 (1984) 151–159.
[6] C.D. Godsil, Algebraic Combinatorics, Chapman and Hall, New York, London, 1993.
[7] C.R. Johnson, B.D. Sutton, Hermitian matrices, eigenvalue multiplicities, and eigenvector components, SIAM J. Matrix Anal. Appl. 26 (2) (2004) 390–399.
[8] I.-J. Kim, B.L. Shader, Nonsingular acyclic matrices, Linear Multilinear Algebra 57 (4) (2009) 399–407.
[9] I.-J. Kim, B.L. Shader, On Fiedler- and Parter-vertices of acyclic matrices, Linear Algebra Appl. 428 (11–12) (2008) 2601–2613.
[10] I.-J. Kim, B.L. Shader, Classification of trees each of whose associated acyclic matrices with distinct diagonal entries has distinct eigenvalues, Bull. Korean Math. Soc. 45 (1) (2008) 95–99.