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Various Series Related to the Polylogarithmic Function

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Abstract: We derive some series related to the polylogarithmic function, and we also give a new proof to the existing series. Our approach is based on using the summation and integral representation methods. We obtain various interesting series as a consequence.

Keywords: polylogarithmic function; series; harmonic numbers; integration

MSC: 11B75; 11Y35

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1. Introduction and Preliminaries

Series and the evaluation of them trace back to ancient times, where people realised the pattern for summing the first n numbers. Many great names we know today that concerned themselves with the evaluation of those series followed, such as Euler [1] and Malmsten [2]. More about series and their evaluations can be found in [3–6]. With the passing of time and experience growing, so did the knowledge of many mathematical functions. The one we will discuss and use in this paper is the polylogarithmic function. It was defined in many ways by multiple authors, but the definition we use here is the standard one today. First systematization of the polylogarithm as a function was done by Lewin [7]. Multiple polylogarithm and hyperlogarithm functions were introduced by Lappo-Danilevski in 1927 [8] and subsequently rediscovered when evaluating Feynman loop integrals in four dimensions, which is particularly useful when devising algorithms to numerically compute some vacuum integrals (see, for instance, [9,10] and references therein). We will express our integral representations in terms of hyperlogarithm when possible. This paper is an extension of the results found in [6,11,12].

To start with, we explain the notation we will use throughout the paper. The first known definition is as follows.

Definition 1. *The polylogarithm is defined by a power series in z, given by [7]*

$$\mathrm{Li}_s(z) = \sum_{k=1}^{+\infty} \frac{z^k}{k^s}.$$

This definition is valid for arbitrary complex order s and for all complex arguments z with $|z| < 1$. *We will also need the definition given by*

$$\mathrm{Li}_s(z) = \int_0^z \frac{\mathrm{Li}_{s-1}(z)}{z} dz.$$

The iterative definition via integrals starts with

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\ln(1-z)}{z} dz.$$

For $z = 1$ *we get the Riemann zeta function* ζ*, which is also a function of complex variable s. For more information, see [13,14].*

$$\mathrm{Li}_s(1) = \zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \Re(s) > 1$$

The second definition is as follows.

Definition 2. *The harmonic numbers are defined as follows [15]:*

$$H_n := 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$$

for $n \geqslant 1$ *and by definition* $H_0 = 0$

Definition 3. *The skew harmonic numbers are defined as follows [15]:*

$$\overline{H_n} = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}$$

The easily obtainable integral representation:

$$\overline{H_n} = \ln 2 - \int_0^1 \frac{(-u)^n}{1+u} du$$

Definition 4. *The generalized hypergeometric function* ${}_qF_q(a; b; x)$ *is defined as follows [16]:*

$${}_pF_q(a; b; x) = \sum_{k=0}^{+\infty} \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \frac{x^k}{k!} \tag{1}$$

where $(a)_k$ *is the Pochhammer symbol defined as [16]*

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = a(a+1)\ldots(a+k-1). \tag{2}$$

Definition 5. *The multiple polylogarithm is defined as [8]*

$$\mathrm{Li}_{s_1,\ldots,s_m}(z_1,\ldots,z_m) = \sum_{k_1>k_2>\cdots>k_m>0}^{+\infty} \left(\frac{z_1^{k_1}}{k_1^{s_1}}\right) \ldots \left(\frac{z_m^{k_m}}{k_m^{s_m}}\right) \tag{3}$$

Definition 6. *The hyperlogarithm is defined as follows [8]:*

$$\mathrm{G}(z_1,\ldots,z_k; y) = \int_0^y \frac{dt_1}{t_1 - z_1} \int_0^{t_2} \frac{dt_2}{t_2 - z_2} \cdots \int_0^{t_{k-1}} \frac{dt_k}{t_k - z_k} \tag{4}$$

for all $z_i \neq 0, i = 1 \dots k$. *If* $z_i = 0$ *for all* $i = 1, \dots, k$ *then*

$$G(\overbrace{0, \dots, 0}^{k}; y) = \frac{1}{k!}(\ln y)^k. \tag{5}$$

If only j *of them,* $j < k$*, are trailing zeroes*

$$G(z_1, \dots, z_{k-j}, \overbrace{0, \dots, 0}^{j}; y) \tag{6}$$

it can be shown that there are recursive procedures to eliminate all the existing singularities $\ln y$ *for* $y = 0$ *[9].*

Using the notation

$$\mathrm{G}_{s_1,\dots,s_m}(z_1, \dots, z_m; y) = \mathrm{G}(\overbrace{0, \dots, 0}^{s_1-1}, z_1, \dots, z_{m-1}, \overbrace{0, \dots, 0}^{s_m-1}, z_m; y)$$

it is possible to express (3) in terms of (4) as

$$\mathrm{Li}_{s_1,\dots,s_m}(z_1, \dots, z_m) = (-1)^m \, \mathrm{G}_{s_1,\dots,s_m}\left(\frac{1}{z_1}, \frac{1}{z_1 \cdot z_2}, \dots, \frac{1}{z_1 \cdot \dots z_m}; 1\right) \tag{7}$$

2. Main Results

In our analysis we will need Abel's summation formula [17], which states that if $(a_n)_{n \geqslant 1}$ and $(b_n)_{n \geqslant 1}$ are two sequences of real numbers and $A_n = \sum_{k=1}^{n} a_k$, then

$$\sum_{k=1}^{n} a_k b_k = A_n b_{n+1} + \sum_{k=1}^{n} A_k (b_k - b_{k+1}).$$

We will also be using, in our calculations, the infinite version of the preceding formula:

$$\sum_{k=1}^{+\infty} a_k b_k = \lim_{n \to +\infty} (A_n b_{n+1}) + \sum_{k=1}^{+\infty} A_k (b_k - b_{k+1}).$$

We will need some lemmas in order to proceed further. The following lemma will be extensively used throughout the paper.

Lemma 1. *The following equality holds for* $|z| < 1$ *[6]:*

$$\sum_{k=1}^{+\infty} \frac{z^{k+1}}{(k+1)^{s-1}} = \mathrm{Li}_{s-1}(z) - z$$

We will also need the following Lemma.

Lemma 2. *The following equality holds [12]:*

$$\int_0^x \frac{t^n}{1-t} dt = \ln\left(\frac{1}{1-x}\right) - x - \dots - \frac{x^n}{n}$$

Lemma 3. *The following equality holds for any q and for $|z| < 1$ [6]:*

$$\lim_{k\to+\infty} k^q \left(\mathrm{Li}_s(z) - z - \cdots - \frac{z^{k+1}}{(k+1)^s} \right) = 0.$$

Lemma 4. *Let $\mathrm{Li}_s(z)$ denote the polylogarithmic function. The following equality holds for $|z| < 1$ and $|y| < 1$:*

$$\sum_{k=1}^{+\infty} y^k \left(\mathrm{Li}_s(z) - z - \cdots - \frac{z^k}{k^s} \right) = \frac{1}{y-1}(\mathrm{Li}_s(zy) - zy) - \frac{y}{y-1}(\mathrm{Li}_s(z) - z)$$

Proof. We will use Abel's summation formula.
Choosing $a_k = y^k, b_k = \mathrm{Li}_s(z) - z - \cdots - \frac{z^k}{k^s}$. We proceed as follows.

$$\sum_{k=1}^{+\infty} y^k \left(\mathrm{Li}_s(z) - z - \cdots - \frac{z^k}{k^s} \right)$$

$$= \lim_{k\to+\infty} \frac{y(y^k-1)}{y-1} \left(\mathrm{Li}_s(z) - z - \cdots - \frac{z^{k+1}}{(k+1)^s} \right) + \sum_{k=1}^{+\infty} \left(\frac{y^{k+1}}{y-1} - \frac{y}{y-1} \right) \frac{z^{k+1}}{(k+1)^s}$$

The limit goes to zero due to Lemma 3. One obtains

$$\frac{1}{y-1} \sum_{k=1}^{+\infty} \frac{(yz)^{k+1}}{(k+1)^s} - \frac{y}{y-1} \sum_{k=1}^{+\infty} \frac{z^{k+1}}{(k+1)^s}.$$

Using Lemma 1 one obtains

$$\frac{1}{y-1}(\mathrm{Li}_s(zy) - zy) - \frac{y}{y-1}(\mathrm{Li}_s(z) - z).$$

□

Theorem 1. *The following equality holds for $|z| < 1, |y| < 1$:*

$$\sum_{k=1}^{+\infty} y^k H_k \left(\mathrm{Li}_s(z) - z - \cdots - \frac{z^k}{k^s} \right) =$$
$$\int_0^1 \left(\frac{1}{y-1}(\mathrm{Li}_s(zy) - zy) - \frac{y}{y-1}(\mathrm{Li}_s(z) - z) - \right.$$
$$\left. \left(\frac{1}{yu-1}(\mathrm{Li}_s(zyu) - zyu) - \frac{yu}{yu-1}(\mathrm{Li}_s(z) - z) \right) \right) \frac{du}{1-u}.$$

Proof. We begin with rewriting the harmonic series as an integral:

$$\sum_{k=1}^{+\infty} y^k H_k \left(\mathrm{Li}_s(z) - z - \cdots - \frac{z^k}{k^s} \right) = \sum_{k=1}^{+\infty} y^k \left(\mathrm{Li}_s(z) - z - \cdots - \frac{z^k}{k^s} \right) \int_0^1 \frac{1-u^k}{1-u} du =$$

$$\int_0^1 \frac{\sum_{k=1}^{+\infty} y^k \left(\mathrm{Li}_s(z) - z - \cdots - \frac{z^k}{k^s} \right) - \sum_{k=1}^{+\infty} (uy)^k \left(\mathrm{Li}_s(z) - z - \cdots - \frac{z^k}{k^s} \right)}{1-u} du\ .$$

For both sums, one can use Lemma 4, directly for the first one and taking yu for y for the second one. The result reads

$$\sum_{k=1}^{+\infty} y^k H_k \left(\mathrm{Li}_s(z) - z - \cdots - \frac{z^k}{k^s} \right) =$$

$$\int_0^1 \frac{\frac{1}{y-1}(\mathrm{Li}_s(zy) - zy) - \frac{y}{y-1}(\mathrm{Li}_s(z) - z) - \left(\frac{1}{yu-1}(\mathrm{Li}_s(zyu) - zyu) - \frac{yu}{yu-1}(\mathrm{Li}_s(z) - z) \right)}{1-u} du.$$

□

Corollary 1. *The following equalities come from Theorem 1.*

$$a) \sum_{k=1}^{+\infty} y^k H_k \left(\mathrm{Li}_1(z) - z - \cdots - \frac{z^k}{k} \right) = \frac{\mathrm{Li}_2\left(\frac{z}{z-1}\right) - \mathrm{Li}_2\left(\frac{z(1-y)}{z-1}\right) - \mathrm{Li}_2\left(\frac{yz}{yz-1}\right)}{y-1}$$

Setting $z = \frac{1}{2}, y = \frac{1}{3}$ *we get*

$$b) \sum_{k=1}^{+\infty} \left(\frac{1}{3}\right)^k H_k \left(\mathrm{Li}_1\left(\frac{1}{2}\right) - \frac{1}{2} - \cdots - \frac{(\frac{1}{2})^k}{k} \right) = \frac{\pi^2}{8} + \frac{3}{2}\left(\mathrm{Li}_2\left(-\frac{2}{3}\right) + \mathrm{Li}_2\left(-\frac{1}{5}\right) \right)$$

Proof. Part (a): setting $s = 1$ in the previously derived theorem it can be shown that

$$\sum_{k=1}^{+\infty} y^k H_k \left(\mathrm{Li}_1(z) - z - \cdots - \frac{z^k}{k} \right) = \frac{\mathrm{Li}_2\left(\frac{z}{z-1}\right) - \mathrm{Li}_2\left(\frac{z(1-y)}{z-1}\right) - \mathrm{Li}_2\left(\frac{yz}{yz-1}\right)}{y-1}.$$

Setting $y = \frac{1}{3}, x = \frac{1}{2}$ we arrive at part (b). □

Theorem 2. *The following equality holds for* $|z| < 1, |y| < 1$.

$$\begin{aligned} &\sum_{k=1}^{+\infty} y^k \overline{H_k} \left(\mathrm{Li}_s(z) - z - \cdots - \frac{z^k}{k^s} \right) \\ &= \ln 2 \left(\frac{1}{y-1}(\mathrm{Li}_s(zy) - zy) - \frac{y}{y-1}(\mathrm{Li}_s(z) - z) \right) \\ &- \int_0^1 \left(-\frac{1}{uy+1}(\mathrm{Li}_s(-zyu) + zyu) - \frac{yu}{1+yu}(\mathrm{Li}_s(z) - z) \right) \frac{du}{1+u} \end{aligned}$$

Proof. Similar to the last theorem, rewriting skew harmonic numbers as an integral, the result follows. □

Corollary 2. *The following equality comes from Theorem 2.*

$$\sum_{k=1}^{+\infty} y^k \overline{H_k} \left(\mathrm{Li}_1(z) - z - \cdots - \frac{z^k}{k} \right)$$

$$= \frac{\mathrm{Li}_2\left(\frac{(y+1)z}{z-1}\right) + \mathrm{Li}_2\left(\frac{yz}{yz-1}\right) - \mathrm{Li}_2\left(\frac{2yz}{yz-1}\right) - \mathrm{Li}_2\left(\frac{z}{z-1}\right)}{y-1}.$$

Proof. Setting $s = 1$ in the previously derived theorem, it can be shown that

$$\sum_{k=1}^{+\infty} y^k \overline{H_k}\left(\mathrm{Li}_1(z) - z - \cdots - \frac{z^k}{k}\right)$$
$$= \frac{\mathrm{Li}_2\left(\frac{(y+1)z}{z-1}\right) + \mathrm{Li}_2\left(\frac{yz}{yz-1}\right) - \mathrm{Li}_2\left(\frac{2yz}{yz-1}\right) - \mathrm{Li}_2\left(\frac{z}{z-1}\right)}{y-1}.$$

□

Theorem 3. *The following equality holds for* $|x| < 1$ *and* $|z| < 1$.

$$\sum_{k=1}^{+\infty} \left(\ln\left(\frac{1}{1-x}\right) - x - \cdots - \frac{x^k}{k}\right)\left(\mathrm{Li}_s(z) - z - \cdots - \frac{z^k}{k^s}\right)$$
$$= \int_0^x (-(\mathrm{Li}_s(tz) - tz) + t(\mathrm{Li}_s(z) - z))\frac{dt}{(1-t)^2}$$

Proof. We will use Lemma 2 to rewrite the first brackets as an integral. Interchanging the sum and the integral thanks to Fubini's theorem, we get

$$\sum_{k=1}^{+\infty} \int_0^x \frac{t^k}{1-t}dt\left(\mathrm{Li}_s(z) - z - \cdots - \frac{z^k}{k^s}\right) = \int_0^x \frac{\sum_{k=1}^{+\infty} t^k\left(\mathrm{Li}_s(z) - z - \cdots - \frac{z^k}{k^s}\right)}{1-t}dt.$$

Using Lemma 4 for the sum inside the integral, we obtain the equality

$$\sum_{k=1}^{+\infty} \left(\ln\left(\frac{1}{1-x}\right) - x - \cdots - \frac{x^k}{k}\right)\left(\mathrm{Li}_s(z) - z - \cdots - \frac{z^k}{k^s}\right)$$
$$= \int_0^x (-(\mathrm{Li}_s(tz) - tz) + t(\mathrm{Li}_s(z) - z))\frac{dt}{(1-t)^2}$$

An alternative representation of the integral could be given in terms of the hyperlogarithm (4):

$$\int_0^x (-(\mathrm{Li}_s(tz) - tz) + t(\mathrm{Li}_s(z) - z))\frac{dt}{(1-t)^2}$$
$$= \left(\frac{x}{1-x} + \ln(1-x)\right)\mathrm{Li}_s(z) - \frac{x}{1-x}\mathrm{Li}_s(xz) + G\left(1, 0, \ldots, 0, \frac{1}{z}, x\right)$$

with $s-2$ zeros. □

Corollary 3. *The following equality comes from Theorem 3.*

$$a)\ \sum_{k=1}^{+\infty} \left(\ln\left(\frac{1}{1-x}\right) - x - \cdots - \frac{x^k}{k}\right)\left(\mathrm{Li}_1(z) - z - \cdots - \frac{z^k}{k^1}\right)$$
$$= -\frac{\ln(1-zx)}{x-1} + \ln(1-z)\left(\frac{1}{x-1} - \ln(1-x)\right)$$
$$-z\left(\frac{\ln(1-xz)}{z-1} - \frac{\ln(1-x)}{z-1}\right) + \ln(1-z).$$

Proof. Setting $s = 1$ in Theorem 3, it can be shown that

$$\sum_{k=1}^{+\infty}\left(\ln\left(\frac{1}{1-x}\right) - x - \cdots - \frac{x^k}{k}\right)\left(\mathrm{Li}_1(z) - z - \cdots - \frac{z^k}{k^1}\right) =$$

$$-\frac{\ln(1-zx)}{x-1} + \ln(1-z)\left(\frac{1}{x-1} - \ln(1-x)\right) - z\left(\frac{\ln(1-tz)}{z-1} - \frac{\ln(1-x)}{z-1}\right) + \ln(1-z).$$

□

Corollary 4. *The following equality comes from Theorem 3.*

$$\sum_{k=1}^{+\infty}\left(\ln\left(\frac{1}{1-x}\right) - x - \cdots - \frac{x^k}{k}\right)^2$$
$$= -x\left(\frac{\ln(1-x^2)}{x-1} - \frac{\ln(1-x)}{x-1}\right) - \frac{\ln(1-x^2)}{x-1}$$
$$+ \ln(1-x)\left(\frac{1}{x-1} - \ln(1-x)\right) + \ln(1-x).$$

This result can be found in [12] derived in a different way.

Proof. Setting $z = x$ in Theorem 3 and realising that $\mathrm{Li}_1(x) = \ln\left(\frac{1}{1-x}\right)$, we get the result found in [12] derived in a different way:

$$\sum_{k=1}^{+\infty}\left(\ln\left(\frac{1}{1-x}\right) - x - \cdots - \frac{x^k}{k}\right)^2 =$$

$$-x\left(\frac{\ln(1-x^2)}{x-1} - \frac{\ln(1-x)}{x-1}\right) - \frac{\ln(1-x^2)}{x-1} + \ln(1-x)\left(\frac{1}{x-1} - \ln(1-x)\right) + \ln(1-x).$$

Setting $x = \frac{1}{2}$ we get

$$\sum_{k=1}^{+\infty}\left(\ln\left(\frac{1}{1-\frac{1}{2}}\right) - \frac{1}{2} - \cdots - \frac{(\frac{1}{2})^k}{k}\right)^2 = \ln\left(\frac{27}{16}\right) - \ln^2 2.$$

□

Corollary 5. *The following equality comes from Theorem 3.*

$$\sum_{k=1}^{+\infty}\left(\ln\left(\frac{1}{1-x}\right) - x - \cdots - \frac{x^k}{k}\right)\left(\mathrm{Li}_2(z) - z - \cdots - \frac{z^k}{k^2}\right)$$
$$= z\left(\frac{\ln(|x|\ln(1-zx) + \mathrm{Li}_2(zx)}{z} - \frac{\ln(1-x)\ln(1+\frac{zx-z}{z-1}) + \mathrm{Li}_2(-\frac{zx-z}{z-1})}{z}\right)$$
$$+(\ln(1-x) - \ln(|x|))\ln(1-zx) + \frac{\mathrm{Li}_2(zx)}{x-1} - \mathrm{Li}_2(z)\left(\frac{1}{x-1} - \ln(1-x)\right)$$
$$+\mathrm{Li}_2\left(\frac{z}{z-1}\right) - \mathrm{Li}_2(z).$$

Proof. Setting $s = 2$ in the previously derived theorem, it can be shown that

$$\sum_{k=1}^{+\infty}\left(\ln\left(\frac{1}{1-x}\right) - x - \cdots - \frac{x^k}{k}\right)\left(\mathrm{Li}_2(z) - z - \cdots - \frac{z^k}{k^2}\right)$$

$$= z\left(\frac{\ln(|x|\ln(1-zx) + \mathrm{Li}_2(zx)}{z} - \frac{\ln(1-x)\ln(1+\frac{zx-z}{z-1}) + \mathrm{Li}_2(-\frac{zx-z}{z-1})}{z}\right)$$

$$+(\ln(1-x) - \ln(|x|))\ln(1-zx) + \frac{\mathrm{Li}_2(zx)}{x-1} - \mathrm{Li}_2(z)\left(\frac{1}{x-1} - \ln(1-x)\right)$$

$$+\mathrm{Li}_2\left(\frac{z}{z-1}\right) - \mathrm{Li}_2(z).$$

Setting $z = \frac{1}{5}$ and $x = \frac{1}{3}$ we get

$$\sum_{k=1}^{+\infty}\left(\ln\left(\frac{1}{1-\frac{1}{3}}\right) - \frac{1}{3} - \cdots - \frac{(\frac{1}{3})^k}{k}\right)\left(\mathrm{Li}_2\left(\frac{1}{5}\right) - \frac{1}{5} - \cdots - \frac{(\frac{1}{5})^k}{k^2}\right)$$

$$= \mathrm{Li}_2\left(-\frac{1}{4}\right) - \mathrm{Li}_2\left(\frac{1}{5}\right) - \frac{3\,\mathrm{Li}_2\left(\frac{1}{15}\right)}{2} - \mathrm{Li}_2\left(\frac{1}{5}\right)\left(\ln\left(\frac{3}{2}\right) - \frac{3}{2}\right)$$

$$+\frac{1}{5}\left(5\left(\mathrm{Li}_2\left(\frac{1}{15}\right) + \ln(3)\ln\left(\frac{15}{14}\right)\right) - 5\left(\mathrm{Li}_2\left(-\frac{1}{6}\right) - \ln\left(\frac{7}{6}\right)\ln\left(\frac{3}{2}\right)\right)\right)$$

$$+\left(\ln(3) - \ln\left(\frac{3}{2}\right)\right)\left(-\ln\left(\frac{15}{14}\right)\right) \sim 0.000810864.$$

□

Theorem 4. *The following equality holds for* $|z| < 1, |y| < 1, |x| < 1$.

$$\sum_{k=1}^{+\infty} y^k\left(\ln\left(\frac{1}{1-x}\right) - x - \cdots - \frac{x^k}{k}\right)\left(\mathrm{Li}_s(z) - z - \cdots - \frac{z^k}{k^s}\right)$$
$$= \int_0^x\left(\frac{1}{ty-1}(\mathrm{Li}_s(tyz) - tyz) - \frac{ty}{ty-1}(\mathrm{Li}_s(z) - z)\right)\frac{dt}{1-t}$$

Proof. The proof follows similarly to the previously derived theorem.
The integral representation in terms of the hyperlogarithm (4) is given by

$$\int_0^x\left(\frac{1}{ty-1}(\mathrm{Li}_s(tyz) - tyz) - \frac{ty}{ty-1}(\mathrm{Li}_s(z) - z)\right)\frac{dt}{1-t}$$

$$= -\frac{\mathrm{Li}_s(xyz)(\ln(1-x) - \ln(1-xy))}{y-1} + \frac{\left(\ln(1-x)\,\mathrm{Li}_s(xyz) + G\left(1,0,\ldots,0,\frac{1}{zy},x\right)\right)}{y-1}$$

$$-\frac{\left(\ln(1-xy)\,\mathrm{Li}_s(xyz) - G\left(1,0,\ldots,0,\frac{1}{z},xy\right)\right)}{y-1} + \frac{\mathrm{Li}_s(z)(y\ln(1-x) - \ln(1-xy))}{y-1}$$

where the number of zeros is $s-1$. □

Corollary 6. *The following equalities come from Theorem 4.*

$$a)\ \sum_{k=1}^{+\infty} y^k\left(\ln\left(\frac{1}{1-x}\right) - x - \cdots - \frac{x^k}{k}\right)\left(\mathrm{Li}_1(z) - z - \cdots - \frac{z^k}{k^1}\right)$$

$$= \frac{\mathrm{Li}_2\left(\frac{-z(yx-1)}{z-1}\right) - \mathrm{Li}_2\left(\frac{-yz(x-1)}{yz-1}\right) + (\ln(1-yz) - y\ln(1-z))\ln(1-x)}{y-1}$$
$$-\frac{\mathrm{Li}_2\left(\frac{z}{z-1}\right) - \mathrm{Li}_2\left(\frac{yz}{yz-1}\right)}{y-1}.$$

Proof. Setting $s = 1$ in the previously derived theorem, it can be shown that the following holds:

$$\sum_{k=1}^{+\infty} y^k \left(\ln\left(\frac{1}{1-x}\right) - x - \cdots - \frac{x^k}{k}\right)\left(\mathrm{Li}_1(z) - z - \cdots - \frac{z^k}{k^1}\right)$$
$$= \frac{\mathrm{Li}_2\left(\frac{-z(yx-1)}{z-1}\right) - \mathrm{Li}_2\left(\frac{-yz(x-1)}{yz-1}\right) + (\ln(1-yz) - y\ln(1-z))\ln(1-x)}{y-1}$$
$$-\frac{\mathrm{Li}_2\left(\frac{z}{z-1}\right) - \mathrm{Li}_2\left(\frac{yz}{yz-1}\right)}{y-1}.$$

Setting $x, y, z = \frac{1}{2}$ we get

$$\sum_{k=1}^{+\infty} \left(\frac{1}{2}\right)^k \left(\ln\left(\frac{1}{1-\frac{1}{2}}\right) - \frac{1}{2} - \cdots - \frac{(\frac{1}{2})^k}{k}\right)^2$$
$$= 2\left(-\mathrm{Li}_2\left(-\frac{1}{3}\right) - \frac{\pi^2}{12}\right)$$
$$-2\left(\mathrm{Li}_2\left(-\frac{3}{4}\right) - \mathrm{Li}_2\left(-\frac{1}{6}\right) - \left(\frac{\ln(2)}{2} - \ln\left(\frac{4}{3}\right)\right)\ln(2)\right) \sim 0.0199097.$$

□

Theorem 5. *The following equality holds for* $|x| < 1$ *and* $|z| < 1$.

$$\sum_{k=1}^{+\infty} H_k \left(\ln\left(\frac{1}{1-x}\right) - x - \cdots - \frac{x^k}{k}\right)\left(\mathrm{Li}_s(z) - z - \cdots - \frac{z^s}{k^s}\right)$$
$$= \int_0^x \int_0^1 \left(\frac{1}{t-1}(\mathrm{Li}_s(zt) - zt) - \frac{t}{t-1}(\mathrm{Li}_s(z) - z) - \right.$$
$$\left.\left(\frac{1}{tu-1}(\mathrm{Li}_s(ztu) - ztu) - \frac{tu}{tu-1}(\mathrm{Li}_s(z) - z)\right)\right)\frac{du}{1-u}\frac{dt}{1-t}$$

Proof. Using Lemma 2 to rewrite the first brackets and using Theorem 1, which states

$$\sum_{k=1}^{+\infty} y^k H_k \left(\mathrm{Li}_s(z) - z - \ldots - \frac{z^k}{k^s}\right) =$$

$$\int_0^1 \frac{\frac{1}{y-1}(\mathrm{Li}_s(zy) - zy) - \frac{y}{y-1}(\mathrm{Li}_s(z) - z) - \left(\frac{1}{yu-1}(\mathrm{Li}_s(zyu) - zyu) - \frac{yu}{yu-1}(\mathrm{Li}_s(z) - z)\right)}{1-u} du$$

we get the result. □

Corollary 7. *The following equalities come from Theorem 5.*

$$\sum_{k=1}^{+\infty} H_k\left(\ln\left(\frac{1}{1-x}\right)-x-\cdots-\frac{x^k}{k}\right)\left(\mathrm{Li}_1(z)-z-\cdots-\frac{z^k}{k^1}\right)$$
$$=-z\left(-\frac{\ln(1-x)\ln(1+\frac{zx-z}{z-1})+\mathrm{Li}_2(-\frac{zx-z}{z-1})}{z^2-z}+\frac{\ln^2(1-zx)}{2z-2}\right.$$
$$\left.-\frac{\ln(|x|)\ln(1-zx)+\mathrm{Li}_2(zx)}{z}\right)$$
$$=-\frac{(1-zx)\ln(1-zx)+(z-1)\,\mathrm{Li}_2(-\frac{zx-z}{z-1})+\ln(1-z)(z-1)}{(z-1)x-z+1}$$
$$-\ln(1-zx)\left(-\frac{z\ln(1-zx)}{z-1}+\ln(|x|)+\frac{\ln(1-x)}{z-1}\right)-\frac{\mathrm{Li}_2(\frac{zx}{zx-1})}{x-1}+\frac{\mathrm{Li}_2(\frac{z}{z-1})}{x-1}$$
$$-\frac{z\ln(1-x)}{z-1}+z\left(-\frac{\mathrm{Li}_2(\frac{z}{z-1})}{z^2-z}\right)+\frac{(z-1)\,\mathrm{Li}_2(\frac{z}{z-1})+\ln(1-z)(z-1)}{1-z}+\mathrm{Li}_2\left(\frac{z}{z-1}\right).$$

Proof. Setting $s=1$ in the previous theorem and using Corollary 1 gives us

$$\sum_{k=1}^{+\infty} H_k\left(\ln\left(\frac{1}{1-x}\right)-x-\cdots-\frac{x^k}{k}\right)\left(\mathrm{Li}_1(z)-z-\cdots-\frac{z^k}{k^1}\right)$$
$$=\int_0^x \frac{\mathrm{Li}_2\left(\frac{z}{z-1}\right)-\mathrm{Li}_2\left(\frac{z(1-t)}{z-1}\right)-\mathrm{Li}_2\left(\frac{zt}{zt-1}\right)}{t-1}\cdot\frac{1}{1-t}dt=$$
$$\left(z\left(\frac{\ln(|t-1|)\ln\left(\left|\frac{zt-z}{z-1}+1\right|\right)+\mathrm{Li}_2\left(-\frac{zt-z}{z-1}\right)}{z^2-z}-\frac{\ln^2(|zt-1|)}{2z-2}+\frac{\ln(|t|)\ln(|zt-1|)+\mathrm{Li}_2(zt)}{z}\right)\right.$$
$$-\frac{(1-zt)\ln(|zt-1|)+(z-1)\,\mathrm{Li}_2\left(-\frac{zt-z}{z-1}\right)+\ln(z-1)(z-1)}{(z-1)t-z+1}$$
$$-\ln(|zt-1|)\left(-\frac{z\ln(|zt-1|)}{z-1}+\ln(|t|)+\frac{\ln(|t-1|)}{z-1}\right)$$
$$\left.-\frac{\mathrm{Li}_2\left(\frac{zt}{zt-1}\right)}{t-1}+\frac{\mathrm{Li}_2\left(\frac{z}{z-1}\right)}{t-1}-\frac{z\ln(|t-1|)}{z-1}\right)\Bigg|_0^x=$$
$$-z\left(-\frac{\ln(1-x)\ln(1+\frac{zx-z}{z-1})+\mathrm{Li}_2\left(-\frac{zx-z}{z-1}\right)}{z^2-z}+\frac{\ln^2(1-zx)}{2z-2}-\frac{\ln(|x|)\ln(1-zx)+\mathrm{Li}_2(zx)}{z}\right)$$
$$-\frac{(1-zx)\ln(1-zx)+(z-1)\,\mathrm{Li}_2\left(-\frac{zx-z}{z-1}\right)+\ln(1-z)(z-1)}{(z-1)x-z+1}$$
$$-\ln(1-zx)\left(-\frac{z\ln(1-zx)}{z-1}+\ln(|x|)+\frac{\ln(1-x)}{z-1}\right)-\frac{\mathrm{Li}_2\left(\frac{zx}{zx-1}\right)}{x-1}+\frac{\mathrm{Li}_2(\frac{z}{z-1})}{x-1}$$
$$-\frac{z\ln(1-x)}{z-1}+z\left(-\frac{\mathrm{Li}_2\left(\frac{z}{z-1}\right)}{z^2-z}\right)+\frac{(z-1)\,\mathrm{Li}_2\left(\frac{z}{z-1}\right)+\ln(1-z)(z-1)}{1-z}+\mathrm{Li}_2\left(\frac{z}{z-1}\right).$$

Setting $x=z=\frac{1}{2}$ we get

$$\sum_{k=1}^{+\infty} H_k\left(\ln\left(\frac{1}{1-\frac{1}{2}}\right)-\frac{1}{2}-\cdots-\frac{(\frac{1}{2})^k}{k}\right)^2\sim 0.0458638.$$

□

The most interesting result, in our opinion, is Theorem 6 containing a squared bracket, and Corollary 8 part b with brackets of the third degree.

Theorem 6. *The following equality holds for* $|x| < 1$ *and* $|z| < 1$.

$$\sum_{k=1}^{+\infty}\left(\ln\frac{1}{1-x}-x-\cdots-\frac{x^k}{k}\right)^2\left(\mathrm{Li}_s(z)-z-\cdots-\frac{z^k}{k^s}\right)$$

$$=\int_0^x\int_0^x\left(\frac{1}{ut-1}(\mathrm{Li}_s(utz)-utz)-\frac{ut}{ut-1}(\mathrm{Li}_s(z)-z)\right)\frac{du}{1-u}\frac{dt}{1-t}.$$

Proof. We begin by rewriting the first brackets as the integral using Lemma 2, then we exchange the places of the sum and the integral. We then proceed using Theorem 4.

The hyperharmonic representation is the same as the one found in Theorem 4, just taking its integral. □

Corollary 8. *The following equalities come from Theorem 6.*

$$(a)\sum_{k=1}^{+\infty}\left(\ln\frac{1}{1-x}-x-\cdots-\frac{x^k}{k}\right)^2\left(\mathrm{Li}_1(z)-z-\cdots-\frac{z^k}{k^1}\right)$$

$$=\frac{z(\ln(1-zx)\ln(1+\frac{xzx-x}{x-1})+\mathrm{Li}_2(-\frac{xzx-x}{x-1}))}{z-1}$$

$$+\frac{(xz-1)(\ln(1-xzx)\ln(1-\frac{xzx-1}{xz-1})+\mathrm{Li}_2(\frac{xzx-1}{xz-1}))}{(x-1)z-x+1}$$

$$-\frac{x(\ln(1-xzx)\ln(1-\frac{xzx-1}{z-1})+\mathrm{Li}_2(\frac{xzx-1}{z-1}))}{x-1}$$

$$-\frac{(2zx-2z)\ln(1-zx)\ln(1-xzx)+(2z-2)\mathrm{Li}_2(-\frac{(x-1)zx}{zx-1})+(2-2z)\mathrm{Li}_2(\frac{zx}{zx-1})}{(2z-2)x-2z+2}$$

$$-\frac{((2-2z)\mathrm{Li}_2(-\frac{xzx-z}{z-1})+(zx-z)\ln^2(1-zx)+(z-zx)\ln^2(1-zx))}{(2z-2)x-2z+2}$$

$$+\ln(|x||z||x|)\ln(1-xzx)+\frac{x\ln(1-z)\ln(1-x\cdot x)}{x-1}+\mathrm{Li}_2(1-xzx)$$

$$+\ln(1-x)\left(z\left(\frac{\ln(1-zx)}{z-1}-\frac{\ln(1-x)}{z-1}\right)+\frac{\ln(1-zx)}{x-1}\right)$$

$$-\ln(1-x)\ln(1-z)(\frac{1}{x-1}-\ln(1-x)-\frac{\mathrm{Li}_2(\frac{z}{z-1})}{x-1}-\frac{x\ln(1-z)\ln(1-x)}{x-1}$$

$$-z\left(\frac{\mathrm{Li}_2(\frac{x}{x-1})}{z-1}\right)-\frac{\mathrm{Li}_2(\frac{-1}{xz-1})}{(x-1)z-x+1}+\frac{x\,\mathrm{Li}_2(\frac{-1}{z-1})}{x-1}+$$

$$+\frac{(2-2z)\mathrm{Li}_2(\frac{z}{z-1})}{2-2z}-\mathrm{Li}_2(1)-\ln(1-x)\ln(1-z)-\mathrm{Li}_2\left(\frac{z}{z-1}\right).$$

(b) Setting $z=x$ *we get the following sum:*

$$\sum_{k=1}^{+\infty}\left(\ln\left(\frac{1}{1-x}\right)-x-\cdots-\frac{x^k}{k}\right)^3.$$

Setting $x=z=\frac{1}{2}$ *we get*

$$\sum_{k=1}^{+\infty}\left(\ln\left(\frac{1}{1-\frac{1}{2}}\right)-\frac{1}{2}-\cdots-\frac{(\frac{1}{2})^k}{k}\right)^3\sim 0.00754194894828.$$

Proof. Part (a): setting $s = 1$ in the last theorem and using the corollary of Theorem 4 for the integral inside, we arrive at the expression

$$\sum_{k=1}^{+\infty}\left(\ln\frac{1}{1-x}-x-\cdots-\frac{x^k}{k}\right)^2\left(\mathrm{Li}_1(z)-z-\cdots-\frac{z^k}{k^1}\right)$$
$$=\int_0^x\int_0^x\left(\frac{1}{ut-1}(\mathrm{Li}_1(utz)-utz)-\frac{ut}{ut-1}(\mathrm{Li}_1(z)-z)\right)\frac{du}{1-u}\frac{dt}{1-t}$$
$$=\int_0^x\left(\mathrm{Li}_2\left(\frac{-z(tx-1)}{z-1}\right)-\mathrm{Li}_2\left(\frac{-tz(x-1)}{tz-1}\right)+(\ln(1-tz)-t\ln(1-z))\ln(1-x)-\right.$$
$$\left.\mathrm{Li}_2\left(\frac{z}{z-1}\right)-\mathrm{Li}_2\left(\frac{tz}{tz-1}\right)\right)\frac{dt}{(1-t)^2}$$
$$=\frac{z(\ln(1-zx)\ln(1+\frac{xzx-x}{x-1})+\mathrm{Li}_2(-\frac{xzx-x}{x-1}))}{z-1}$$
$$+\frac{(xz-1)(\ln(1-xzx)\ln(1-\frac{xzx-1}{xz-1})+\mathrm{Li}_2(\frac{xzx-1}{xz-1}))}{(x-1)z-x+1}$$
$$-\frac{x(\ln(1-xzx)\ln(1-\frac{xzx-1}{z-1})+\mathrm{Li}_2(\frac{xzx-1}{z-1}))}{x-1}$$
$$-\frac{(2zx-2z)\ln(1-zx)\ln(1-xzx)+(2z-2)\mathrm{Li}_2(-\frac{(x-1)zx}{zx-1})+(2-2z)\mathrm{Li}_2(\frac{zx}{zx-1})}{(2z-2)x-2z+2}$$
$$-\frac{((2-2z)\mathrm{Li}_2(-\frac{xzx-z}{z-1})+(zx-z)\ln^2(1-zx)+(z-zx)\ln^2(1-zx))}{(2z-2)x-2z+2}$$
$$+\ln(|x||z||x|)\ln(1-xzx)+\frac{x\ln(1-z)\ln(1-x\cdot x)}{x-1}+\mathrm{Li}_2(1-xzx)$$
$$+\ln(1-x)\left(z\left(\frac{\ln(1-zx)}{z-1}-\frac{\ln(1-x)}{z-1}\right)+\frac{\ln(1-zx)}{x-1}\right)$$
$$-\ln(1-x)\ln(1-z)(\frac{1}{x-1}-\ln(1-x)-\frac{\mathrm{Li}_2(\frac{z}{z-1})}{x-1}-\frac{x\ln(1-z)\ln(1-x)}{x-1}$$
$$-z\left(\frac{\mathrm{Li}_2(\frac{x}{x-1})}{z-1}\right)-\frac{\mathrm{Li}_2(\frac{-1}{xz-1})}{(x-1)z-x+1}+\frac{x\,\mathrm{Li}_2(\frac{-1}{z-1})}{x-1}$$
$$+\frac{(2-2z)\mathrm{Li}_2(\frac{z}{z-1})}{2-2z}-\mathrm{Li}_2(1)-\ln(1-x)\ln(1-z)-\mathrm{Li}_2(\frac{z}{z-1})$$

Part (b): setting $z = x = \frac{1}{2}$ we get the desired result. □

Theorem 7. *The following equality holds for* $|x| < 1$*:*

$$\sum_{k=1}^{+\infty}\left(\ln\left(\frac{1}{1-x}\right)-x-\cdots-\frac{x^k}{k}\right)^3$$
$$=\int_0^x\int_0^x\int_0^x\frac{tzl}{(1-t)(1-z)(1-l)(1-tzl)}dtdzdl$$

Proof. Using Lemma 2 three times for the expression

$$\sum_{k=1}^{+\infty}\left(\ln\left(\frac{1}{1-x}\right)-x-\cdots-\frac{x^k}{k}\right)^3$$

we get

$$\sum_{k=1}^{+\infty}\left(\ln\left(\frac{1}{1-x}\right)-x-\cdots-\frac{x^k}{k}\right)^3=\sum_{k=1}^{+\infty}\int_0^x\frac{t^k}{1-t}dt\int_0^x\frac{z^k}{1-z}dz\int_0^x\frac{l^k}{1-l}dl$$
$$=\int_0^x\int_0^x\int_0^x\frac{\sum_{k=1}^{\infty}(tzl)^k}{(1-t)(1-z)(1-l)}dtdzdl$$
$$=\int_0^x\int_0^x\int_0^x\frac{tzl}{(1-t)(1-z)(1-l)(1-tzl)}dtdzdl.$$

□

Corollary 9. *The following equality holds:*

$$\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\frac{tyz}{(1-t)(1-y)(1-z)(1-tyz)}dtdydz\sim 0.00754194894828.$$

Proof. Setting $x=\frac{1}{2}$ in Theorem 7 and using Corollary 8, part b, the result follows. □

A hypothesis of the harmonic sum:

$$f(x,q)=\sum_{k=1}^{+\infty}H_k\left(\frac{x^{k+1}}{(k+1)(1-x)}\right)^q \tag{8}$$

First five powers of $f(x,q), q=1\ldots5$:

$$f(x,1)=\frac{x\log^2(1-x)}{2x-2x^2},$$
$$f(x,2)=-\frac{x^2\,{}_4F_3^{(\{0,0,0,0\},\{0,0,1\},0)}(\{1,1,1,1\},\{2,2,1\},x^2)}{(x-1)^2},$$
$$f(x,3)=\frac{x^3\,{}_5F_4^{(\{0,0,0,0,0\},\{0,0,0,1\},0)}(\{1,1,1,1,1\},\{2,2,2,1\},x^3)}{(x-1)^3},$$
$$f(x,4)=-\frac{x^4\,{}_6F_5^{(\{0,0,0,0,0,0\},\{0,0,0,0,1\},0)}(\{1,1,1,1,1,1\},\{2,2,2,2,1\},x^4)}{(x-1)^4},$$
$$f(x,5)=\frac{x^5\,{}_7F_6^{(\{0,0,0,0,0,0,0\},\{0,0,0,0,0,1\},0)}(\{1,1,1,1,1,1,1\},\{2,2,2,2,2,1\},x^5)}{(x-1)^5}$$

Notation: the numbers in the power mean the qth derivative of the corresponding variable.

$$\begin{aligned}&{}_5F_4^{(\{0,0,0,0,0\},\{0,0,0,1\},0)}\left(\{1,1,1,1,1\},\{2,2,2,1\},x^3\right)\\&=\frac{\partial}{\partial b_4}\,{}_5F_4(a;b;x^3)\Big|_{a=(1,1,1,1,1),b=(2,2,2,1)}\end{aligned} \tag{9}$$

Hypothesis for $q > 2$, verified up to $q = 60$:

$$f(x,q) = (-1)^{q+1} \frac{x^q}{(x-1)^q} \times$$
$$\frac{\partial}{\partial b_q} \left[{}_{q+2}F_{q+1} \left(\overbrace{1,\ldots,1}^{(q+2)\text{ times}} ; \overbrace{2,\ldots,2}^{q\text{ times}}, 1; x^q \right) \right] \quad (10)$$

3. Conclusions and Outlook

1. In this paper we found more series of the form found in [6,12]. We also gave a new proof of the squared identity found in the new book [12] while also giving new series as well as giving the summation of the expression raised to the third power. Moreover, novel results obtained have been rewritten in terms of a hyperlogarithmic function when possible.
2. To assure accuracy of the results, we verified all the series identities through Wolfram Alpha [18].
3. Further questions can be asked regarding the series of the form $(f(x))^q$ for $q \in \mathbb{N}$, for which functions can the integral representations be found.

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