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NEW RESULTS FOR A CLASS OF UNIVALENT FUNCTIONS*

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Abstract Let $\mathcal{A}$ denote the family of all analytic functions $f(z)$ in the unit disk $\mathbb{D}=\{z\in\mathbb{C}:\ |z|<1\}$, normalized by the conditions $f(0)=0$ and $f'(0)=1$. Let $\mathcal{U}$ denote the set of all functions $f\in\mathcal{A}$ satisfying the condition

$$\left|\left(\frac{z}{f(z)}\right)^2 f'(z)-1\right|<1 \quad \text{for } z\in\mathbb{D}.$$

Let Ω be the class of all $f\in\mathcal{A}$ for which

$$|zf'(z)-f(z)|<\frac{1}{2},\ z\in\mathbb{D}.$$

In this paper, the relations between the two classes are discussed. Furthermore, some new results on the class Ω are obtained.

Key words analytic; univalent; coefficient; Hadamard product

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1 Introduction

Let $\mathcal{A}$ denote the family of all analytic functions $f(z)$ in the unit disk $\mathbb{D}:=\{z\in\mathbb{C}:\ |z|<1\}$, normalized by the conditions $f(0)=0$ and $f'(0)=1$. Denote by $\mathcal{S}$ the subset of $\mathcal{A}$ which consists of univalent functions. Let $\mathcal{S}^*$ and $\mathcal{K}$ denote the subclasses of $\mathcal{S}$ which are starlike and convex in $\mathbb{D}$, respectively, and let $\mathcal{U}$ denote the set of all $f\in\mathcal{A}$ satisfying the condition

$$\left|\left(\frac{z}{f(z)}\right)^2 f'(z)-1\right|<1 \quad \text{for } z\in\mathbb{D}.$$

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It is well known that $\mathcal{U}$ is a subclass of $\mathcal{S}$ [1]. In recent years, many scholars have studied the properties of the family $\mathcal{U}$ [2–6].

In a recent paper, Peng and Zhong [7] introduced the class Ω which consists of functions f in $\mathcal{A}$ satisfying the condition

$$|zf'(z)-f(z)|<\frac{1}{2},\ z\in\mathbb{D}. \tag{1.1}$$

Also, the authors showed that (1.1) is equivalent with

$$f(z)=z+\frac{1}{2}z\int_0^z\varphi(\zeta)\mathrm{d}\zeta, \tag{1.2}$$

where φ is analytic in $\mathbb{D}$ and $|\varphi(z)|\le 1,\ z\in\mathbb{D}$. We note that in the same paper it is proved that $\Omega\subset\mathcal{S}^*$.

In this paper we discuss the relations between $\mathcal{U}$ and Ω. Also, we consider the other properties of the class Ω and get some new results.

2 Relations Between $\mathcal{U}$ and Ω

Theorem 2.1 The class Ω is not a subset of the class $\mathcal{U}$.

Proof Let us consider the function

$$\varphi_1(z)=\frac{z+a}{1+az},\ 0<a<1.$$

Then $\varphi_1:\mathbb{D}\to\mathbb{D}$, and the appropriate function $f_1\in\Omega$ given by (1.2) has the form

$$f_1(z)=z+\frac{1}{2}z\int_0^z\frac{\zeta+a}{1+a\zeta}\mathrm{d}\zeta=z+\frac{1}{2a}z^2-\frac{1-a^2}{2a^2}z\log(1+az).$$

From above we have

$$f_1'(z)=1+\frac{1}{a}z-\frac{1-a^2}{2a^2}\log(1+az)-\frac{1-a^2}{2a}\frac{z}{1+az},$$

and so,

$$\left|\left(\frac{z}{f_1(z)}\right)^2f_1'(z)-1\right|_{z=-1}=\left|\frac{2a^2\left(3a^2-a-(1-a^2)\log(1-a)\right)}{\left(2a^2-a-(1-a^2)\log(1-a)\right)^2}-1\right|\to 3$$

when $a\to 1$. It means that for the points in $\mathbb{D}$ near to the point $z=-1$ and for a close to 1 we have

$$\left|\left(\frac{z}{f_1(z)}\right)^2f_1'(z)-1\right|>1.$$

This implies that $f_1\notin\mathcal{U}$. □

Theorem 2.2 If $f\in\Omega$, then $f\in\mathcal{U}$ in the disc $|z|<\sqrt{\frac{\sqrt{5}-1}{2}}=0.78615\cdots$.

Proof If $f\in\Omega$, then we have the representation (1.2). If we put $\omega(z)=\int_0^z\varphi(\zeta)\mathrm{d}\zeta$, then $|\omega(z)|\le|z|,\ |\omega'(z)|\le 1$ and

$$f(z)=z+\frac{1}{2}z\omega(z). \tag{2.1}$$

By using a result of Dieudonné ([8], pp.198–199), we have the next inequality

$$|z\omega'(z)-\omega(z)|\le\frac{r^2-|\omega(z)|^2}{1-r^2}, \tag{2.2}$$

where $|z| = r$ and $|\omega(z)| \le r$. It follows from (2.1) and (2.2) that

$$\begin{aligned}\left|\left(\frac{z}{f(z)}\right)^2 f'(z) - 1\right| &= \left|\frac{z}{f(z)} - z\left(\frac{z}{f(z)}\right)' - 1\right| \\ &= \left|\frac{\frac{1}{2}(z\omega'(z) - \omega(z)) - \frac{1}{4}\omega^2(z)}{\left(1 + \frac{1}{2}\omega(z)\right)^2}\right| \\ &\le \frac{\frac{1}{2}|z\omega'(z) - \omega(z)| + \frac{1}{4}|\omega(z)|^2}{\left(1 - \frac{1}{2}|\omega(z)|\right)^2} \\ &\le \frac{\frac{1}{2}\frac{r^2 - |\omega(z)|^2}{1 - r^2} + \frac{1}{4}|\omega(z)|^2}{\left(1 - \frac{1}{2}|\omega(z)|\right)^2}.\end{aligned}$$

If

$$\frac{\frac{1}{2}\frac{r^2 - |\omega(z)|^2}{1 - r^2} + \frac{1}{4}|\omega(z)|^2}{\left(1 - \frac{1}{2}|\omega(z)|\right)^2} < 1, \tag{2.3}$$

then we have

$$\left|\left(\frac{z}{f(z)}\right)^2 f'(z) - 1\right| < 1.$$

But the inequality (2.3) is equivalent to

$$|\omega(z)|^2 - 2(1 - r^2)|\omega(z)| + 2 - 3r^2 > 0. \tag{2.4}$$

Noting that $|\omega(z)| \le |z| = r$, if we put $|\omega(z)| = t$, with $0 \le t \le r$, and consider the function

$$F(t) = t^2 - 2(1 - r^2)t + 2 - 3r^2,$$

then it is an elementary fact to show that the function F is positive for $0 \le r < r_0 = \sqrt{\frac{\sqrt{5}-1}{2}}$, that is, the inequality (2.4) holds when $|z| < r_0$. And therefore, f is in $\mathcal{U}$ in the disc $|z| < r_0$.

□

3 Estimation of Coefficients

Definition 3.1 ([8], p.151) The logarithmic coefficients γ_n of f in $\mathcal{S}$ is defined by

$$\log \frac{f(z)}{z} = 2\sum_{n=1}^{\infty} \gamma_n z^n, |z| < 1.$$

Theorem 3.2 Let $f \in \Omega$ and let $\gamma_1, \gamma_2, \gamma_3$ be its logarithmic coefficients. Then

(a) $|\gamma_1| \le \frac{1}{4}$;

(b) $|\gamma_2| \le \frac{1}{8}$;

(c) $|\gamma_3| \le \frac{1}{12}$.

All results are the best possible.

Proof We will use the representation (2.1). If we put $\omega(z) = c_1 z + c_2 z^2 + \cdots$, then from $|\omega'(z)| = |c_1 + 2c_2 z + 3c_3 z^2 + \cdots| \le 1$, we have

$$|c_1| \le 1,\ |2c_2| \le 1 - |c_1|^2,\ |3c_3| \le 1 - |c_1|^2 - \frac{4|c_2|^2}{1 + |c_1|} \tag{3.1}$$

(see Prokhorov and Szinal [9]). By using (2.1) we have

$$\begin{aligned}\log\frac{f(z)}{z} &= \log\left(1+\frac{1}{2}\omega(z)\right)\\ &= \log\left(1+\frac{1}{2}(c_1z+c_2z^2+\cdots)\right)\\ &= \frac{1}{2}c_1z+\frac{1}{2}\left(c_2-\frac{1}{4}c_1^2\right)z^2+\frac{1}{2}\left(c_3-\frac{1}{2}c_1c_2+\frac{1}{12}c_1^3\right)z^3+\cdots,\end{aligned}$$

which implies that

$$2\gamma_1=\frac{1}{2}c_1,\ 2\gamma_2=\frac{1}{2}\left(c_2-\frac{1}{4}c_1^2\right),\ 2\gamma_3=\frac{1}{2}\left(c_3-\frac{1}{2}c_1c_2+\frac{1}{12}c_1^3\right). \tag{3.2}$$

Combining (3.1) with (3.2), we have

$$|\gamma_1|=\frac{1}{4}|c_1|\le\frac{1}{4},\ |\gamma_2|\le\frac{1}{8}(2|c_2|+\frac{1}{2}|c_1|^2)\le\frac{1}{8}.$$

Similarly,

$$\begin{aligned}12|\gamma_3| &= \left|3c_3-\frac{3}{2}c_1c_2+\frac{1}{4}c_1^3\right|\\ &\le 3|c_3|+\frac{3}{2}|c_1||c_2|+\frac{1}{4}|c_1|^3\\ &\le 1-|c_1|^2-\frac{4|c_2|^2}{1+|c_1|}+\frac{3}{2}|c_1||c_2|+\frac{1}{4}|c_1|^3\\ &= \psi(|c_1|,|c_2|),\end{aligned}$$

where

$$\psi(x,y)=1-x^2-\frac{4y^2}{1+x}+\frac{3}{2}xy+\frac{1}{4}x^3,(x,y)\in D$$

and D is defined by the conditions: $0\le x\le 1,\ 0\le y\le 1,\ y\le\frac{1}{2}(1-x^2)$. It is easy to check that the function ψ has only one critical point $(0,0)$ belonging to the boundary of the domain D and that $\psi(x,y)\le 1$ in the domain D. This implies that $|\gamma_3|\le\frac{1}{12}$. If we choose the function φ in (1.2) to be $1, z, z^2$ respectively, then we obtain that all results in this theorem are sharp. □

Theorem 3.3 If $f(z)=z+\sum\limits_{n=1}^{\infty}a_nz^n\in\Omega$ and if the inverse function of f has an expansion

$$f^{-1}(w)=w+A_2w^2+A_3w^3+A_4w^4+\cdots \tag{3.3}$$

near $w=0$, then

$$|A_2|\le\frac{1}{2},\ |A_3|\le\frac{1}{2},\ |A_4|\le\frac{5}{8}.$$

All these results are the best possible.

Proof By using the identity $f(f^{-1})=w$ and the representations for the functions f and f^{-1}, we can obtain the next relations

$$\begin{cases}A_2=-a_2,\\ A_3=-a_3+2a_2^2,\\ A_4=-a_4+5a_2a_3-5a_2^3.\end{cases} \tag{3.4}$$

On the other hand, in view of (2.1), if we put $\omega(z) = c_1 z + c_2 z^2 + \cdots$, where $|\omega(z)| \le |z|$, $|\omega'(z)| \le 1$, we have

$$f(z) = z + \sum_{n=2}^{\infty} \frac{1}{2} c_{n-1} z^n. \tag{3.5}$$

Combining (3.4) with (3.5), we obtain

$$\begin{cases} A_2 = -\frac{1}{2}c_1, \\ A_3 = -\frac{1}{2}c_2 + \frac{1}{2}c_1^2, \\ A_4 = -\frac{1}{2}c_3 + \frac{5}{4}c_1c_2 - \frac{5}{8}c_1^3. \end{cases} \tag{3.6}$$

From (3.6) it follows that $|A_2| = \frac{1}{2}|c_1| \le \frac{1}{2}$. Also, by using (3.6) and (3.1), we have

$$|A_3| \le \frac{1}{2}|c_2| + \frac{1}{2}|c_1|^2 \le \frac{1}{4}(1 - |c_1|^2) + \frac{1}{2}|c_1|^2 \le \frac{1}{4} + \frac{1}{4}|c_1|^2 \le \frac{1}{2}.$$

Finally, from (3.6), we obtain that

$$|A_4| = \frac{1}{2}\left|c_3 - \frac{5}{2}c_1c_2 + \frac{5}{4}c_1^3\right| \le \frac{1}{2} \cdot \frac{5}{4} = \frac{5}{8}$$

by using the result of Prokhorov and Szinal (with $\mu = -\frac{5}{2}$ and $\nu = \frac{5}{2}$)[9]. If we consider the function $w = f(z) = z + \frac{1}{2}z^2$, then we have that

$$z = f^{-1}(w) = -1 + \sqrt{1 + 2w} = w - \frac{1}{2}w^2 + \frac{1}{2}w^3 - \frac{5}{8}w^4 + \cdots,$$

which means that our results are the best possible. □

Theorem 3.4 Let $f \in \Omega$ and let γ_n, $n = 1, 2, 3, \cdots$, be its logarithmic coefficients. Then

(a) $\sum_{n=1}^{\infty} |\gamma_n|^2 \le \frac{1}{4} Li_2\left(\frac{1}{4}\right)$, where $\frac{1}{4} Li_2\left(\frac{1}{4}\right) = \sum_{n=1}^{\infty} \frac{1}{n^2}\left(\frac{1}{4}\right)^{n+1}$ is the best possible;

(b) $\sum_{n=1}^{\infty} n^2 |\gamma_n|^2 \le \frac{1}{4}$;

(c) $|\gamma_n| \le \frac{1}{2n}$, $n = 1, 2, \cdots$.

Proof (a) If $f \in \Omega$, then from (2.1) we have

$$f(z) = z + \frac{1}{2} z\omega(z),$$

where $|\omega(z) \le |z|$ and $|\omega'(z)| \le 1$. From here we have

$$\frac{f(z)}{z} \prec 1 + \frac{1}{2}z,$$

which implies

$$\log \frac{f(z)}{z} \prec \log\left(1 + \frac{1}{2}z\right),$$

or

$$\sum_{n=1}^{\infty} 2\gamma_n z^n \prec \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n2^n} z^n.$$

By using Rogosinsky's result([8], p.192) we obtain

$$\sum_{n=1}^{\infty} 4|\gamma_n|^2 \le \sum_{n=1}^{\infty} \frac{1}{n^2 2^{2n}} = \sum_{n=1}^{\infty} \frac{\left(\frac{1}{4}\right)^n}{n^2} = Li_2\left(\frac{1}{4}\right).$$

From the last equality we have the statement (a) of the theorem. The function $f(z)=z+\frac{1}{2}z^2$ shows that our result is the best possible.

(b) By using the representation (2.1) and the facts for the function ω, we have

$$\log\frac{f(z)}{z}=\log\left(1+\frac{1}{2}\omega(z)\right). \tag{3.7}$$

From (3.7), after derivation, we get

$$\left(\log\frac{f(z)}{z}\right)'=\frac{\frac{1}{2}\omega'(z)}{1+\frac{1}{2}\omega(z)}. \tag{3.8}$$

Noting that $|\omega(z)|\le 1$ and $|\omega'(z)|\le 1$, from (3.8) we have that

$$\left|\sum_{n=1}^{\infty}2n\gamma_n z^{n-1}\right|\le\frac{\frac{1}{2}|\omega'(z)|}{1-\frac{1}{2}|\omega(z)|}<1. \tag{3.9}$$

The last relation (with $|z|=r$) gives

$$\sum_{n=1}^{\infty}4n^2|\gamma_n|^2r^{2(n-1)}<1. \tag{3.10}$$

Letting r tend to 1 in (3.10), we have the statement (b) of the theorem.

(c) From (b) of this theorem we have $n^2|\gamma_n|^2\le\frac{1}{4}$, which implies $|\gamma_n|\le\frac{1}{2n}$, $n=1,2,\cdots$.

□

Remark 3.5 If we compare the result (c) of Theorem 3.4 with the results of of Theorem 3.2, we conclude that it is not the best possible. We conjecture that $|\gamma_n|\le\frac{1}{4n}$ for $n=1,2,\cdots$. But we don't know how to prove it.

4 Robinson's 1/2-Conjecture and 1/2 Theorem on the Class Ω

Theorem 4.1 Robinson's 1/2-conjecture is valid for the class Ω, i.e., if $f\in\Omega$, then the function

$$F(z)=\frac{1}{2}(f(z)+zf'(z)) \tag{4.1}$$

is univalent in the disc $|z|<\frac{1}{2}$.

Proof If $f\in\Omega$, then by (2.1) we have

$$f(z)=z+\frac{1}{2}z\omega(z),$$

where $|\omega(z)|\le|z|$ and $|\omega'(z)|\le 1$ for $z\in\mathbb{D}$. From here we have that the function F defined by (4.1) is equal to

$$F(z)=z+\frac{1}{2}z(\omega(z)+\frac{1}{2}z\omega'(z))=z+\frac{3}{4}z\omega_1(z),$$

where

$$\omega_1(z)=\frac{2}{3}(\omega(z)+\frac{1}{2}z\omega'(z)).$$

Since $\omega_1(0)=0$ and

$$|\omega_1(z)|\le\frac{2}{3}(|\omega(z)|+\frac{1}{2}|z||\omega'(z)|)\le\frac{2}{3}(|z|+\frac{1}{2}|z|)<1,\ z\in\mathbb{D},$$

it follows that $|\omega_1(z)| \le |z| < \frac{1}{2}$ for $|z| = r < \frac{1}{2}$. Also, by the result of Dieudonné, we have

$$|z\omega_1'(z) - \omega_1(z)| \le \frac{r^2 - |\omega_1(z)|^2}{1-r^2} \le \frac{r^2}{1-r^2} < \frac{1}{3}$$

for $|z| = r < \frac{1}{2}$. By using all these facts, we finally have

$$\begin{aligned}\left|\frac{zF'(z)}{F(z)} - 1\right| &= \left|\frac{\frac{3}{4}z\omega_1'(z)}{1+\frac{3}{4}\omega_1(z)}\right| = \left|\frac{\frac{3}{4}(z\omega_1'(z) - \omega_1(z)) + \frac{3}{4}\omega_1(z)}{1+\frac{3}{4}\omega_1(z)}\right| \\ &\le \frac{\frac{3}{4}|z\omega_1'(z) - \omega_1(z)| + \frac{3}{4}|\omega_1(z)|}{1-\frac{3}{4}|\omega_1(z)|} < \frac{\frac{3}{4}\cdot\frac{1}{3} + \frac{3}{4}\cdot\frac{1}{2}}{1-\frac{3}{4}\cdot\frac{1}{2}} = 1\end{aligned}$$

for $|z| = r < \frac{1}{2}$, which implies that the function F is starlike in the disc $|z| < \frac{1}{2}$. □

Theorem 4.2 If $f \in \Omega$, then

$$|f'(z) - 1| < 1, \ z \in \mathbb{D}.$$

Proof From the representation (2.1), we have

$$f'(z) = 1 + \frac{1}{2}\left(\omega(z) + z\omega'(z)\right)$$

and it follows that

$$|f'(z) - 1| \le \frac{1}{2}\left(|\omega(z)| + |z||\omega'(z)|\right) \le |z| < 1.$$

□

Theorem 4.3 If $f \in \Omega$, then the range of f contains the disk $\{w : |w| < \frac{1}{2}\}$. The number $\frac{1}{2}$ is the best possible.

Proof If $f \in \Omega$, then by the results in [7], we have $f \in S^\star$ and

$$|f(z)| \ge |z| - \frac{1}{2}|z|^2. \tag{4.2}$$

Let $\mathbb{D}_r = \{z \in \mathbb{C} : |z| \le r\}$ for $0 \le r < 1$. Since f is univalent on $\mathbb{D}_r$ and the image of the circle $|z| = r$ under f is a Jordan curve Γ_r, $f(\mathbb{D}_r)$ is a closed domain bounded by Γ_r. Noting the inequality (4.2), $f(\mathbb{D}_r)$ contains a closed disk $\{w : |w| \le r - \frac{r^2}{2}\}$. Since $\mathbb{D} = \bigcup_{0\le r<1} \mathbb{D}_r$, $f(\mathbb{D}) = \bigcup_{0\le r<1} f(\mathbb{D}_r) \supset \{w : |w| < \frac{1}{2}\}$.

If considering the function $f(z) = z + \frac{1}{2}z^2 \in \Omega$, we know that the number $\frac{1}{2}$ is the best possible. □

5 Libera Integral Operator

Libera [10] introduced the integral operator

$$L(f) = \frac{2}{z}\int_0^z f(\zeta)\mathrm{d}\zeta,$$

where $f \in \mathcal{A}$. The Libera integral operator has been studied by several authors on different classes [11–14]. In the paper [10] Libera proved that $L(f) \in \mathcal{K}$ if $f \in \mathcal{K}$ and proved that $L(f) \in \mathcal{C}$ if $f \in \mathcal{C}$, where $\mathcal{K}$ and $\mathcal{C}$ are the class of convex functions and the class of close-to-convex functions respectively. For the class Ω we have the same result.

Theorem 5.1 If $f \in \Omega$, then $L(f) \in \Omega$.

Proof If $f \in \Omega$, then

$$f(z) = z + \frac{1}{2} z \int_0^z \varphi(\zeta)\mathrm{d}\zeta = z + \frac{1}{2} \int_0^1 z^2 \varphi(zt)\mathrm{d}t,$$

where φ is analytic in $\mathbb{D}$ and $|\varphi(z)| \le 1,\ z \in \mathbb{D}$.

$$\begin{aligned} L(f) &= \frac{2}{z} \int_0^z f(\zeta)\mathrm{d}\zeta \\ &= \frac{2}{z} \int_0^z \left(\zeta + \frac{1}{2} \int_0^1 \zeta^2 \varphi(\zeta t)\mathrm{d}t\right) \mathrm{d}\zeta \\ &= z + \frac{1}{2} z^2 \int_0^1 \left(\int_0^1 2\lambda^2 \varphi(z\lambda t)\mathrm{d}t\right) \mathrm{d}\lambda \\ &= z + \frac{1}{2} z^2 \int_0^1 \left(\int_0^1 2\lambda^2 \varphi(z\lambda t)\mathrm{d}\lambda\right) \mathrm{d}t \\ &= z + \frac{1}{2} z^2 \int_0^1 \omega(zt)\mathrm{d}t, \end{aligned}$$

where $\omega(z) = \int_0^1 2\lambda^2 \varphi(z\lambda)\mathrm{d}\lambda$. It is clear that $\omega(z) \in \mathcal{A}$. Since

$$|\omega(z)| = \left| \int_0^1 2\lambda^2 \varphi(z\lambda)\mathrm{d}\lambda \right| \le \int_0^1 2\lambda^2 |\varphi(z\lambda)|\mathrm{d}\lambda \le \int_0^1 2\lambda^2 \mathrm{d}\lambda < 1,$$

we have $L(f) \in \Omega$. □

6 Coefficient Multipliers

The Hadamard product, or convolution, of two power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ and } g(z) = \sum_{n=0}^{\infty} b_n z^n$$

convergent in $\mathbb{D}$ is the function $h = f * g$ with power series

$$h(z) = \sum_{n=0}^{\infty} a_n b_n z^n, |z| < 1.$$

It is clear that

$$h(sz) = \frac{1}{2\pi} \int_0^{2\pi} f(s\mathrm{e}^{\mathrm{i}t}) g(z\mathrm{e}^{-\mathrm{i}t})\mathrm{d}t$$

for $|z| < 1$ and $0 \le s < 1$.

Let $H^p (0 < p \le \infty)$ be the Hardy space consisting of the functions $f \in \mathcal{A}$ which satisfies the condition that $M_p(r, f)$ remains bounded as $r \to 1$, where

$$M_p(r, f) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(r\mathrm{e}^{\mathrm{i}\theta})|^p \mathrm{d}\theta \right\}^{\frac{1}{p}}, \quad 0 < p < \infty$$

and

$$M_\infty(r, f) = \max_{0 \le \theta < 2\pi} |f(r\mathrm{e}^{\mathrm{i}\theta})|.$$

The closed unit ball of H^∞ is denoted by $\mathcal{B}$, that is,

$$\mathcal{B} = \{\varphi(z) : \varphi(z) \in \mathcal{A}, |\varphi(z)| \le 1\}.$$

A complex sequence $\{\lambda_n\}$ is said to be a coefficient multiplier of a family $\mathcal{F}$ of analytic functions into a family $\mathcal{G}$ if $\sum \lambda_n a_n z^n$ belongs to $\mathcal{G}$ for each $f(z) = \sum a_n z^n \in \mathcal{F}$. If we let $g(z) = \sum \lambda_n z^n$, then the sequence$\{\lambda_n\}$ is a coefficient multiplier of $\mathcal{F}$ into $\mathcal{G}$ if and only if $g * f \in \mathcal{G}$ for each $f(z) \in \mathcal{F}$.

Lemma 6.1 If $f \in H^\infty, g \in \mathcal{A}$ and $h = f * g$, then

$$h(z) = \frac{1}{2\pi}\int_0^{2\pi} f(\mathrm{e}^{\mathrm{i}t})g(z\mathrm{e}^{-\mathrm{i}t})\mathrm{d}t.$$

Proof

$$\begin{aligned}\left|h(sz) - \frac{1}{2\pi}\int_0^{2\pi} f(\mathrm{e}^{\mathrm{i}t})g(z\mathrm{e}^{-\mathrm{i}t})\mathrm{d}t\right| &= \left|\frac{1}{2\pi}\int_0^{2\pi}[f(s\mathrm{e}^{\mathrm{i}t}) - f(\mathrm{e}^{\mathrm{i}t})]g(z\mathrm{e}^{-\mathrm{i}t})\mathrm{d}t\right| \\ &\le \frac{1}{2\pi}\int_0^{2\pi}|f(s\mathrm{e}^{\mathrm{i}t}) - f(\mathrm{e}^{\mathrm{i}t})||g(z\mathrm{e}^{-\mathrm{i}t})|\mathrm{d}t \\ &\le \left\{\frac{1}{2\pi}\int_0^{2\pi}|f(s\mathrm{e}^{\mathrm{i}t}) - f(\mathrm{e}^{\mathrm{i}t})|\mathrm{d}t\right\}\max_{|\zeta|=|z|}|g(\zeta)|.\end{aligned}$$

Since $f \in H^\infty \subset H^1$, it follows that([15], p.21)

$$\lim_{s\to 1}\int_0^{2\pi}|f(s\mathrm{e}^{\mathrm{i}t}) - f(\mathrm{e}^{\mathrm{i}t})|\mathrm{d}t = 0.$$

Therefore

$$\left|h(sz) - \frac{1}{2\pi}\int_0^{2\pi} f(\mathrm{e}^{\mathrm{i}t})g(z\mathrm{e}^{-\mathrm{i}t})\mathrm{d}t\right| \to 0$$

as $s \to 1$. This prove that

$$h(z) = \frac{1}{2\pi}\int_0^{2\pi} f(\mathrm{e}^{\mathrm{i}t})g(z\mathrm{e}^{-\mathrm{i}t})\mathrm{d}t.$$

□

Lemma 6.2 Suppose $g \in \mathcal{A}$. Then $g * f \in \mathcal{B}$ for any $f \in \mathcal{B}$ if and only if

$$\min_{h\in H^1}\|g(z\mathrm{e}^{-\mathrm{i}t}) - \mathrm{e}^{\mathrm{i}t}h(\mathrm{e}^{\mathrm{i}t})\|_1 \le 1 \tag{6.1}$$

holds for each $z \in \mathbb{D}$.

Proof For any given $z \in \mathbb{D}$, $g(z/\zeta)/\zeta$ is analytic in the region $\{\zeta : |\zeta| > |z|\}$. So it can define a continuous linear functional on H^∞ as follows:

$$\phi_z(f) = \frac{1}{2\pi\mathrm{i}}\int_{|\zeta|=1} f(\zeta)g(\frac{z}{\zeta})\frac{1}{\zeta}\mathrm{d}\zeta.$$

According to Lemma 6.1, for each $z \in \mathbb{D}$

$$\phi_z(f) = (g * f)(z).$$

Thus, $g * f \in \mathcal{B}$ for any $f \in \mathcal{B}$ if and only if $|\phi_z(f)| \le 1$ for all $f \in \mathcal{B}$ and for each $z \in \mathbb{D}$, or equivalently, if and only if

$$\|\phi_z\| = \sup_{f\in H^\infty, \|f\|_\infty \le 1}|\phi_z(f)| \le 1.$$

Since([15], p.131)

$$\|\phi_z\| = \sup_{f\in H^\infty, \|f\|_\infty\le 1}|\phi_z(f)| = \min_{h\in H^1}\|g(z\mathrm{e}^{-\mathrm{i}t})\mathrm{e}^{-\mathrm{i}t} - h(\mathrm{e}^{\mathrm{i}t})\|_1 = \min_{h\in H^1}\|g(z\mathrm{e}^{-\mathrm{i}t}) - \mathrm{e}^{\mathrm{i}t}h(\mathrm{e}^{\mathrm{i}t})\|_1,$$

we complete the proof of the lemma. □

Theorem 6.3 Suppose $h \in \mathcal{A}$. Then $h*f \in \Omega$ for all $f \in \Omega$ if and only if $h(z) = z+z^2g(z)$, where $g \in \mathcal{A}$ and

$$\min_{h \in H^1} \|g(z\mathrm{e}^{-\mathrm{i}t}) - \mathrm{e}^{\mathrm{i}t}h(\mathrm{e}^{\mathrm{i}t})\|_1 \leq 1$$

holds for each $z \in \mathbb{D}$.

Proof $f \in \Omega$ if and only if there exists a $\varphi \in \mathcal{B}$ such that

$$f(z) = z + \frac{1}{2}z\int_0^z \varphi(\zeta)\mathrm{d}\zeta,$$

or equivalently,

$$f(z) = z + \frac{1}{2}\int_0^1 z^2\varphi(zt)\mathrm{d}t.$$

Since for any $h(z) = z + z^2g(z) \in \mathcal{A}$

$$(h*f)(z) = \big(z + z^2g(z)\big) * \left(z + \frac{1}{2}\int_0^1 z^2\varphi(zt)\mathrm{d}t\right) = z + \frac{1}{2}\int_0^1 z^2(\varphi * g)(zt)\mathrm{d}t,$$

it follows that $h*f \in \Omega$ for all $f \in \Omega$ if and only if $\varphi * g \in \mathcal{B}$ for all $\varphi \in \mathcal{B}$. By Lemma 6.2, we get the conclusion. □

References

[1] Ozaki S, Nunokawa M. The Schwarzian derivative and univalent functions. Proc Amer Math Soc, 1972, **33**: 392–394

[2] Obradović M, Ponnusamy S. Product of univalent functions. Math Comput Modell, 2013, **57**: 793–799

[3] Obradović M, Ponnusamy S. New criteria and distortion theorems for univalent functions. Complex Variables, Theory and Application: An International Journal, 2001, **44**(3): 173–191

[4] Obradović M, Ponnusamy S. Univalence and starlikeness of certain transforms defined by convolution. J Math Anal Appl, 2007, **336**: 758–767

[5] Obradović M, Ponnusamy S. Radius of univalence of certain combination of univalent and analytic functions. Bull Malays Math Sci Soc, 2012, **35**(2): 325–334

[6] Fournier R, Ponnusamy S. A class of locally univalent functions defined by a differential inequality. Complex Variables and Elliptic Equations: An International Journal, 2007, **52**(1): 1–8

[7] Peng Z, Zhong G. Some properties for certain classes of univalent functions defined by differential inequalities. Acta Mathematica Scientia, 2017, **37B**(1): 69–78

[8] Duren P L. Univalent Functions. New York: Springer-Verlag, 1983

[9] Prokhorov D V, Szynal J. Inverse coefficients for (α, β)-convex functions. Ann Univ Mariae Curie-Sklodowska, 1981, **35**: 125–143

[10] Libera R J. Some classes of regular univalent functions. Proc Amer Math Soc, 1965, **16**: 755–758

[11] Nunokawa M, Sokol J, Cho N E, et al. Conditions for starlikeness of the libera operator. J Inequal Appl, 2014, **1**: 1–9

[12] Sokol J. Starlikeness of the Libera transform of functions with bounded turning. Appl Math Comput, 2008, **203**: 273–276

[13] Miller S S, Mocanu P T. Libera transform of functions with bounded turning. J Math Anal Appl 2002, **276**: 90–97

[14] Ponnusamy S. On starlikeness of certain integral transforms. Annales Polonici Mathematici, 1992, **56**(3): 227–232

[15] Duren P L. Theory of H^p Spaces. New York: Academic Press, 1970