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HANKEL DETERMINANT OF TYPE $H_2(3)$ FOR INVERSE FUNCTIONS OF SOME CLASSES OF UNIVALENT FUNCTIONS WITH MISSING SECOND COEFFICIENT

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Abstract. In this paper we determine the upper bounds of $|H_2(3)|$ for the inverse functions of functions of some classes of univalent functions, where $H_2(3)(f) = a_3a_5 - a_4^2$ is the Hankel determinant of a special type.

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1. Introduction and preliminaries

Let $\mathcal{A}$ be the class containing functions that are analytic in the unit disk $\mathbb{D} := \{|z| < 1\}$ and are normalized such that $f(0) = 0 = f'(0) - 1$, i.e.,

$$f(z) = z + a_2z^2 + a_3z^3 + \cdots . \tag{1}$$

By $\mathcal{S}$ we denote the class of functions from $\mathcal{A}$ which are univalent in $\mathbb{D}$.

Also, we need the classes of functions of bounded turning, of convex functions, of starlike functions, and of functions starlike with respect to symmetric points, subclasses of $\mathcal{S}$, defined respectively in the following way

$$\begin{aligned}
\mathcal{R} &= \left[f \in \mathcal{A} : \operatorname{Re} f'(z) > 0,\, z \in \mathbb{D}\right],\\
\mathcal{C} &= \left[f \in \mathcal{A} : \operatorname{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right] > 0,\, z \in \mathbb{D}\right],\\
\mathcal{S}^\star &= \left[f \in \mathcal{A} : \operatorname{Re}\frac{zf'(z)}{f(z)} > 0,\, z \in \mathbb{D}\right],\\
\mathcal{S}^\star_s &= \left[f \in \mathcal{A} : \operatorname{Re}\frac{2zf'(z)}{f(z) - f(-z)} > 0,\, z \in \mathbb{D}\right].
\end{aligned}$$

In his paper [4] Zaprawa considered the Hankel determinant of the type

$$H_2(3)(f) = a_3a_5 - a_4^2,$$

defined for the coefficients of the function f given by (1). The author treated bounds of $|H_2(3)(f)|$ for the classes $\mathcal{R}, \mathcal{C}, \mathcal{S}^\star$ and gave sharp results in the case $a_2 = 0$. He also investigated the general case of these classes. In the same paper it is proved that

$$\max\{|H_2(3)(f)| : f \in \mathcal{S}\} > 1.$$

The object of current paper is to obtained the bounds of the modulus of the Hankel determinant $H_2(3)(f^{-1})$ of coefficients of the inverse of function from the classes $\mathcal{R}$, $\mathcal{C}$, $\mathcal{S}^\star$ and $\mathcal{S}^\star_s$, defined before, as well as for the class $\mathcal{S}$. In all cases we suppose that function f is missing its second coefficient, i.e., $a_2 = 0$.

Namely, for every univalent function in $\mathbb{D}$ exists inverse at least on the disk with radius 1/4 (due to the famous Koebe's 1/4 theorem). If the inverse has an expansion

$$f^{-1}(w) = w + A_2 w^2 + A_3 w^3 + \cdots, \tag{2}$$

then, by using the identity $f(f^{-1}(w)) = w$, from (1) and (2) we receive

$$\begin{aligned} A_2 &= -a_2, \\ A_3 &= -a_3 + 2a_2^2, \\ A_4 &= -a_4 + 5a_2a_3 - 5a_2^3, \\ A_5 &= -a_5 + 6a_2a_4 - 21a_2^2a_3 + 3a_3^2 + 14a_2^4. \end{aligned}$$

Especially, when $a_2 = 0$, we have

$$A_2 = 0, \quad A_3 = -a_3, \quad A_4 = -a_4, \quad A_5 = -a_5 + 3a_3^2.$$

So, in this case,

$$H_2(3)(f^{-1}) = A_3A_5 - A_4^2 = a_3a_5 - a_4^2 - 3a_3^3, \tag{3}$$

i.e.,

$$H_2(3)(f^{-1}) = H_2(3)(f) - 3a_3^3. \tag{4}$$

For our further consideration we need the next lemma given by Carlson [1].

Lemma 1. Let

$$\omega(z) = c_1 z + c_2 z^2 + \cdots \tag{5}$$

be a Schwartz function, i.e., a function analytic in $\mathbb{D}$, $\omega(0) = 0$ and $|\omega(z)| < 1$. Then

$$|c_1| \le 1, \quad |c_2| \le 1 - |c_1|^2, \quad |c_3| \le 1 - |c_1|^2 - \frac{|c_2|^2}{1 + |c_1|}, \quad \text{and} \quad |c_4| \le 1 - |c_1|^2 - |c_2|^2.$$

2. Main results

Theorem 1. *Let $f \in \mathcal{A}$ is given by (1) and let $a_2 = 0$. Then*

(a) $|H_3(2)(f^{-1})| \le \frac{28}{45}$ *if* $f \in \mathcal{R}$;

(b) $|H_3(2)(f^{-1})| \le \frac{2}{45}$ *if* $f \in \mathcal{C}$;

(c) $|H_3(2)(f^{-1})| \le 2$ *if* $f \in \mathcal{S}^\star$;

(d) $|H_3(2)(f^{-1})| \le 2$ *if* $f \in \mathcal{S}^\star_s$.

All these results are sharp.

Proof.

(a) Since $f \in \mathcal{R}$ is equivalent to

$$f'(z) = \frac{1+\omega(z)}{1-\omega(z)},$$

for certain Schwartz function ω, we receive that

$$f'(z) = 1 + 2\omega(z) + 2\omega^2(z) + \cdots . \tag{6}$$

Using the notations for f and ω given by (1) and (5), and equating the coefficients in (6), we receive

$$\begin{cases} a_2 = c_1, \\ a_3 = \frac{2}{3}(c_2 + c_1^2), \\ a_4 = \frac{1}{2}(c_3 + 2c_1c_2 + c_1^3), \\ a_5 = \frac{2}{5}(c_4 + 2c_1c_3 + 3c_1^2c_2 + c_2^2 + c_1^4). \end{cases} \tag{7}$$

Since $a_2 = 0$, by (7) we have $c_1 = 0$, and the appropriate coefficients have the next form:

$$a_3 = \frac{2}{3}c_2, \quad a_4 = \frac{1}{2}c_3, \quad a_5 = \frac{2}{5}(c_4 + c_2^2). \tag{8}$$

Now, from (3) and (8), after simple computation, we obtain

$$H_3(2)(f^{-1}) = \frac{4}{15}c_2c_4 - \frac{1}{4}c_3^2 - \frac{28}{45}c_2^3,$$

and further,

$$|H_3(2)(f^{-1})| \le \frac{4}{15}|c_2||c_4| + \frac{1}{4}|c_3|^2 + \frac{28}{45}|c_2|^3.$$

Applying Lemma 1 (with $c_1 = 0$) we receive

$$|H_3(2)(f^{-1})| \le \frac{4}{15}|c_2|(1-|c_2|^2) + \frac{1}{4}(1-|c_2|^2)^2 + \frac{28}{45}|c_2|^3.$$

and, finally,

$$|H_3(2)(f^{-1})| \le \frac{1}{4} + \frac{4}{15}|c_2| - \frac{1}{2}|c_2|^2 + \frac{16}{45}|c_2|^3 + \frac{1}{4}|c_2|^4 =: \varphi_1(|c_2|), \qquad (9)$$

where $0 \le |c_2| \le 1$. Since

$$\begin{aligned}\varphi_1'(|c_2|) &= \frac{4}{15} - |c_2| + \frac{16}{15}|c_2|^2 + |c_2|^3 \\ &= \frac{4}{15}(1-2|c_2|)^2 + \frac{1}{15}|c_2| + |c_2|^3 > 0,\end{aligned}$$

we have $\varphi_1(|c_2|) \le \varphi_1(1) = \frac{28}{45}$, and from (9),

$$|H_3(2)(f^{-1})| \le \frac{28}{45} = 0.622\ldots.$$

This result is best possible as the function $f_1(z) = \ln\frac{1+z}{1-z} - z$ defined by $f_1'(z) = \frac{1+z^2}{1-z^2}$, shows.

(b) We apply the same method as in the previous case. Namely, from the definition of the class $\mathcal{C}$ we have

$$1 + \frac{zf''(z)}{f'(z)} = \frac{1+\omega(z)}{1-\omega(z)},$$

where ω is a Schwartz function, and from here

$$(zf'(z))' = \left[1 + 2\left(\omega(z) + \omega^2(z) + \cdots\right)\right] \cdot f'(z). \qquad (10)$$

Using the notations (1) and (5), and comparing the coefficients in the relation (10), after some simple calculations, we obtain

$$\begin{cases} a_2 = c_1, \\ a_3 = \frac{1}{3}\left(c_2 + 3c_1^2\right), \\ a_4 = \frac{1}{6}\left(c_3 + 5c_1c_2 + 6c_1^3\right) \\ a_5 = \frac{1}{30}\left(3c_4 + 14c_1c_3 + 43c_1^2c_2 + 30c_1^4 + 6c_2^2\right). \end{cases} \qquad (11)$$

If $a_2 = 0$, then by (11) we have $c_1 = 0$, which implies

$$a_3 = \frac{1}{3}c_2, \quad a_4 = \frac{1}{6}c_3, \quad a_5 = \frac{1}{10}(c_4 + 2c_2^2). \tag{12}$$

Using (3) and (12) we obtain

$$H_3(2)(f^{-1}) = \frac{1}{180}\left(6c_2c_4 - 5c_3^2 - 8c_2^3\right).$$

From the last relation we get

$$|H_3(2)(f^{-1})| \le \frac{1}{180}\left(6|c_2||c_4| + 5|c_3|^2 + 8|c_2|^3\right),$$

and further, after applying Lemma1 (with $c_1 = 0$),

$$|H_3(2)(f^{-1})| \le \frac{1}{180}\left(6|c_2|(1-|c_2|^2) + 5(1-|c_2|^2)^2 + 8|c_2|^3\right),$$

i.e.,

$$|H_3(2)(f^{-1})| \le \frac{1}{180}\left(5 + 6|c_2| - 10|c_2|^2 + 2|c_2|^3 + 5|c_2|^4\right) =: \varphi_2(|c_2|), \tag{13}$$

where $0 \le |c_2| \le 1$. Since

$$\varphi_2'(|c_2|) = \frac{1}{90}\left(3 - 10|c_2| + 3|c_2|^2 + 10|c_2|^3\right),$$

which, after considering this polynomial in the interval $[0,1]$, gives $\varphi_1(|c_2|) \le \varphi_2(1) = \frac{2}{45}$, and further, from (13),

$$|H_3(2)(f^{-1})| \le \frac{2}{45} = 0.044\ldots.$$

The function $f_2(z) = \operatorname{artanh} z$ satisfying $1 + \frac{zf_2''(z)}{f_2'(z)} = \frac{1+z^2}{1-z^2}$ shows that the result is the best possible.

(c) From the definition of the class $\mathcal{S}^\star$ we have that there exists a Schwartz function ω such that

$$\frac{zf'(z)}{f(z)} = \frac{1+\omega(z)}{1-\omega(z)},$$

and from here

$$zf'(z) = \left[1 + 2\left(\omega(z) + \omega^2(z) + \cdots\right)\right] \cdot f(z). \tag{14}$$

As in the two previous cases ((a) and (b)), by comparing the coefficients in the relation (14), and some simple calculations, we have

$$\begin{cases} a_2 = 2c_1 \\ a_3 = c_2 + 3c_1^2 \\ a_4 = \dfrac{2}{3}\left(c_3 + 5c_1c_2 + 6c_1^3\right) \\ a_5 = \dfrac{1}{2}\left(c_4 + \dfrac{14}{3}c_1c_3 + \dfrac{43}{3}c_1^2c_2 + 10c_1^4 + 2c_2^2\right). \end{cases}$$

For the case $a_2 = 0$ we have the next

$$a_3 = c_2, \quad a_4 = \frac{2}{3}c_3, \quad a_5 = \frac{1}{2}(c_4 + 2c_2^2). \tag{15}$$

So, from (3) and (15) we obtain

$$H_3(2)(f^{-1}) = \frac{1}{18}\left(9c_2c_4 - 8c_3^2 - 36c_2^3\right),$$

and from here

$$|H_3(2)(f^{-1})| \le \frac{1}{18}\left(9|c_2||c_4| + 8|c_3|^2 + 36|c_2|^3\right).$$

Using estimates for $|c_4|$ and $|c_3|$ from Lemma1 (with $c_1 = 0$) from the last relation we receive

$$|H_3(2)(f^{-1})| \le \frac{1}{18}\left(8 + 9|c_2| - 16|c_2|^2 + 27|c_2|^3 + 8|c_2|^4\right) =: \varphi_3(|c_2|), \tag{16}$$

where $0 \le |c_2| \le 1$. Since

$$\begin{aligned} \varphi_3'(|c_2|) &= \frac{1}{18}\left(9 - 32|c_2| + 81|c_2|^2 + 32|c_2|^3\right) \\ &= \frac{1}{18}\left[9(1 - 3|c_2|)^2 + 22|c_2| + 32|c_2|^3\right] > 0, \end{aligned}$$

then $\varphi_3(|c_2|) \le \varphi_3(1) = 2$, and from (16),

$$|H_3(2)(f^{-1})| \le 2.$$

The result is the best possible as the function $f_3(z) = \frac{z}{1-z^2}$ shows.

(d) From the definition of the class $\mathcal{S}_s^\star$ we have that there exists a Schwartz function ω such that
$$\frac{2zf'(z)}{f(z)-f(-z)}=\frac{1+\omega(z)}{1-\omega(z)},$$
and from here
$$2zf'(z)=\left[1+2(\omega(z)+\omega^2(z)+\cdots)\right]\cdot[f(z)-f(-z)]. \tag{17}$$
Similarly as in previous cases, by comparing the coefficients in the relation (17), after some simple calculations, we receive
$$\begin{cases} a_2=c_1\\ a_3=c_2+c_1^2\\ a_4=\frac{1}{2}\left(c_3+3c_1c_2+2c_1^3\right)\\ a_5=\frac{1}{2}\left(c_4+2c_1c_3+5c_1^2c_2+2c_1^4+2c_2^2\right). \end{cases} \tag{18}$$
For $a_2=0$ ($c_1=0$), from (18) we get
$$a_3=c_2,\quad a_4=\frac{1}{2}c_3,\quad a_5=\frac{1}{2}(c_4+2c_2^2),$$
and using (3),
$$H_3(2)(f^{-1})=\frac{1}{4}\left(2c_2c_4-c_3^2-8c_2^3\right),$$
and from here
$$|H_3(2)(f^{-1})|\le\frac{1}{4}\left(2|c_2||c_4|+|c_3|^2+8|c_2|^3\right).$$
Using the estimates for $|c_4|$ and $|c_3|$ from Lemma 1 (with $c_1=0$) from the last relation we have
$$|H_3(2)(f^{-1})|\le\frac{1}{4}\left(1+2|c_2|-2|c_2|^2+6|c_2|^3+|c_2|^4\right)=:\varphi_4(|c_2|), \tag{19}$$
where $0\le|c_2|\le1$. Since
$$\begin{aligned}\varphi_4'(|c_2|)&=\frac{1}{2}\left(1-2|c_2|+9|c_2|^2+2|c_2|^3\right)\\ &=\frac{1}{2}\left[(1-|c_2|)^2+8|c_2|+2|c_2|^3\right]>0,\end{aligned}$$

then φ_4 is an increasing function and $\varphi_4(|c_2|) \le \varphi_4(1) = 2$. So, from (19),

$$|H_3(2)(f^{-1})| \le 2.$$

This result is the best possible as the function f_4 defined by

$$\frac{2zf_4'(z)}{f_4(z) - f_4(-z)} = \frac{1+z^2}{1-z^2}$$

shows.

Remark 1. From the relation (4) we get the following.

(a) For $f \in \mathcal{R}$,

$$|H_3(2)(f^{-1}) - H_3(2)(f)| = 3|a_3|^3 = 3\left(\frac{2}{3}|c_2|\right)^3 \le \frac{8}{9},$$

and the result is the best possible as the function f_1 shows (in this case $H_3(2)(f_1) = \frac{4}{15}$ and $H_3(2)(f_1^{-1}) = -\frac{28}{45}$).

(b) For $f \in \mathcal{C}$,

$$|H_3(2)(f^{-1}) - H_3(2)(f)| = 3|a_3|^3 = 3\left(\frac{|c_2|}{3}\right)^3 \le \frac{1}{9},$$

and the result is the best possible as the function f_2 shows.

(c) For $f \in \mathcal{S}^\star$,

$$|H_3(2)(f^{-1}) - H_3(2)(f)| = 3|a_3|^3 = 3|c_2|^3 \le 3,$$

and the result is the best possible as the function f_3 shows.

(d) For $f \in \mathcal{S}_s^\star$,

$$|H_3(2)(f^{-1}) - |H_3(2)(f)| = 3|a_3|^3 = 3|c_2|^3 \le 3,$$

and the result is the best possible for the function f_4.

For obtaining the corresponding result for the whole class $\mathcal{S}$ we will use method based on Grunsky coefficients. In the proof we will use mainly the notations and results given in the book of N.A. Lebedev ([3]).

Here are basic definitions and results.

Let $f\in\mathcal{S}$ and let

$$\log\frac{f(t)-f(z)}{t-z}=\sum_{p,q=0}^{\infty}\omega_{p,q}t^pz^q,$$

where $\omega_{p,q}$ are the Grunsky's coefficients with property $\omega_{p,q}=\omega_{q,p}$. For those coefficients we have the next Grunsky's inequality ([2, 3]):

$$\sum_{q=1}^{\infty}q\left|\sum_{p=1}^{\infty}\omega_{p,q}x_p\right|^2\le\sum_{p=1}^{\infty}\frac{|x_p|^2}{p},\tag{20}$$

where x_p are arbitrary complex numbers such that last series converges.

Further, it is well-known that if the function f given by (1) belongs to $\mathcal{S}$, then also

$$\tilde{f}_2(z)=\sqrt{f(z^2)}=z+c_3z^3+c_5z^5+\cdots\tag{21}$$

belongs to the class $\mathcal{S}$. Then, for the function $\tilde{f}_2$ we have the appropriate Grunsky's coefficients of the form $\omega^{(2)}_{2p-1,2q-1}$ and the inequality (20) has the form:

$$\sum_{q=1}^{\infty}(2q-1)\left|\sum_{p=1}^{\infty}\omega_{2p-1,2q-1}x_{2p-1}\right|^2\le\sum_{p=1}^{\infty}\frac{|x_{2p-1}|^2}{2p-1}.\tag{22}$$

Here, and further in the paper we omit the upper index (2) in $\omega^{(2)}_{2p-1,2q-1}$ if compared with Lebedev's notation.

If in the inequality (22) we put $x_1=1$ and $x_{2p-1}=0$ for $p=2,3,\ldots$, then we receive

$$|\omega_{11}|^2+3|\omega_{13}|^2+5|\omega_{15}|^2+7|\omega_{17}|^2\le 1.\tag{23}$$

As it has been shown in [3, p.57], if f is given by (1) then the coefficients a_2, a_3, a_4 and a_5 are expressed by Grunsky's coefficients $\omega_{2p-1,2q-1}$ of the function $\tilde{f}_2$ given by (21) in the following way:

$$
\begin{aligned}
a_2 &= 2\omega_{11}, \\
a_3 &= 2\omega_{13} + 3\omega_{11}^2, \\
a_4 &= 2\omega_{33} + 8\omega_{11}\omega_{13} + \frac{10}{3}\omega_{11}^3, \\
a_5 &= 2\omega_{35} + 8\omega_{11}\omega_{33} + 5\omega_{13}^2 + 18\omega_{11}^2\omega_{13} + \frac{7}{3}\omega_{11}^4, \\
0 &= 3\omega_{15} - 3\omega_{11}\omega_{13} + \omega_{11}^3 - 3\omega_{33}, \\
0 &= \omega_{17} - \omega_{35} - \omega_{11}\omega_{33} - \omega_{13}^2 + \frac{1}{3}\omega_{11}^4.
\end{aligned}
\tag{24}
$$

We note that in the cited book of Lebedev there exists a typing mistake for the coefficient a_5. Namely, instead of the term $5\omega_{13}^2$, there is $5\omega_{15}^2$.

Theorem 2. *Let $f \in \mathcal{S}$ is given by (1) and let $a_2 = 0$. Then*

$$|H_3(2)(f^{-1})| \le \frac{\sqrt{3}}{6\sqrt{7}} + 2\sqrt{3} = 3.57321\ldots.$$

Proof. In the case when $a_2 = 0$, from (24) we have $\omega_{11} = 0$, and so

$$a_3 = 2\omega_{13}, \quad a_4 = 2\omega_{33}, \quad a_5 = 2\omega_{35} + 5\omega_{13}^2, \quad \omega_{33} = \omega_{15}, \quad \omega_{35} = \omega_{17} - \omega_{13}^2. \tag{25}$$

Using (3) and (25), we have

$$H_3(2)(f^{-1}) = 4\omega_{13}\omega_{35} - 14\omega_{13}^3 - 4\omega_{33}^2,$$

and after applying the two last relations from (25),

$$H_3(2)(f^{-1}) = 4\omega_{13}\omega_{17} - 18\omega_{13}^3 - 4\omega_{15}^2.$$

From here we have

$$|H_3(2)(f^{-1})| \le 4|\omega_{13}||\omega_{17}| + 18|\omega_{13}|^3 + 4|\omega_{15}|^2,$$

or finally, using $|\omega_{17}| \le \frac{1}{\sqrt{7}}\sqrt{1 - 3|\omega_{13}|^2 - 5|\omega_{15}|^2}$ (from (23)) we get

$$
\begin{aligned}
|H_3(2)(f^{-1})| &\le \frac{1}{\sqrt{7}}|\omega_{13}|\sqrt{1 - 3|\omega_{13}|^2 - 5|\omega_{15}|^2} + 18|\omega_{13}|^3 + 4|\omega_{15}|^2 \\
&=: \frac{1}{\sqrt{7}}\psi_1(|\omega_{13}|, |\omega_{15}|) + 2\psi_2(|\omega_{13}|, |\omega_{15}|),
\end{aligned}
\tag{26}
$$

where
$$\psi_1(y,z) = y\sqrt{1-3y^2-5z^2}, \qquad \psi_2(y,z) = 9y^3+2z^2,$$
with $0 \le y = |\omega_{13}| \le \frac{1}{\sqrt{3}}$ and $0 \le z = |\omega_{15}| \le \frac{1}{\sqrt{5}}\sqrt{1-3y^2}$ (where we used the inequality (23)). It is easy to verify that for these range of y and z, $\psi_1(y,z) \le \psi_1(1/\sqrt{6},0) = \frac{\sqrt{3}}{6}$ and $\psi_2(y,z) \le \psi_2(1/\sqrt{3},0) = \sqrt{3}$, so that from (26)) we have
$$|H_3(2)(f^{-1})| \le \frac{\sqrt{3}}{6\sqrt{7}} + 2\sqrt{3} = 3.57321\ldots.$$

Remark 2. From the relation (4) we get for $f \in \mathcal{S}$:
$$|H_3(2)(f^{-1}) - H_3(2)(f)| = 3|a_3|^3 = 3|2\omega_{13}|^3 \le 3\left(2\cdot\frac{1}{\sqrt{3}}\right)^3 = \frac{8}{\sqrt{3}} = 4.6188\ldots.$$

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