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# HANKEL DETERMINANT OF SECOND ORDER FOR INVERSE FUNCTIONS OF CERTAIN CLASSES OF UNIVALENT FUNCTIONS

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ABSTRACT. In this paper we determine mainly sharp upper bounds for the Hankel determinant of second order for the inverse functions of functions from some classes of univalent functions.

## 1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{A}$ be the class containing functions that are analytic in the unit disk $\mathbb{D} := |z| < 1$ and are normalized such that $f(0) = 0 = f'(0) - 1$, i.e.,

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots . \tag{1.1}$$

By $\mathcal{S}$ we denote the class of functions from $\mathcal{A}$ which are univalent in $\mathbb{D}$.

A problem that recently rediscovered, is to find upper bound (preferably sharp) of the modulus of the Hankel determinant $H_q(n)(f)$ of a given function $f$, for $q \geq 1$ and $n \geq 1$, defined by

$$H_q(n)(f) = \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \dots & a_{n+q} \\ \vdots & \vdots & & \vdots \\ a_{n+q-1} & a_{n+q} & \dots & a_{n+2q-2} \end{vmatrix}.$$

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The general Hankel determinant is hard to deal with, so the second and the third ones,

$$H_2(2)(f) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2$$

and

$$H_3(1)(f) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2),$$

respectively, are studied instead. The research is focused on the subclasses of univalent functions (starlike, convex, $\alpha$-convex, close-to-convex, spirallike,...) since the general class of normalised univalent functions is also hard to deal with. Some of the more significant results can be found in [2–6, 8–12, 14].

In this paper we will give mainly sharp upper bound of the modulus of the second Hankel determinant for the inverse functions for the functions in different subclasses of $\mathcal{S}$ as listed bellow.

The classes of convex and starlike functions are defined respectively with

$$\mathcal{C} = \left\{ f \in \mathcal{A} : \ \mathrm{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right] > 0, z \in \mathbb{D} \right\}$$

and

$$\mathcal{S}^* = \left\{ f \in \mathcal{A} : \ \mathrm{Re}\left[\frac{zf'(z)}{f(z)}\right] > 0, z \in \mathbb{D} \right\},$$

where "$\prec$" denotes the usual subordination defined by: $f \prec g$, if, and only if, $f$ and $g$ are analytic in $\mathbb{D}$ and there exists a Schwartz function $\omega$ (analytic in $\mathbb{D}$, $\omega(0) = 0$ and $|\omega(z)| < 1$), such that $f(z) = g(\omega(z))$ for $z \in \mathbb{D}$. If $g$ is univalent, then $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$.

Further, let

$$\mathcal{G} = \left\{ f \in \mathcal{A} : \ \mathrm{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right] < \frac{3}{2}, z \in \mathbb{D} \right\},$$

(that is the subclass of starlike functions, see [7]) and

$$\mathcal{S}^*_q = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec z + \sqrt{1+z^2} \right\}.$$

(also is the subclass of starlike functions).

For every univalent function in $\mathbb{D}$ there exists an inverse one, at least on the disk with radius 1/4 (due to the famous Koebe's 1/4 theorem). If the inverse function has an expansion

$$f^{-1}(w) = w + A_2 w^2 + A_3 w^3 + \cdots , \tag{1.2}$$

then, by using the identity $f(f^{-1}(w)) = w$, from (1.1) and (1.2) we receive

$$\begin{aligned} A_2 &= -a_2, \\ A_3 &= -a_3 + 2a_2^2, \\ A_4 &= -a_4 + 5a_2a_3 - 5a_2^3, \\ A_5 &= -a_5 + 6a_2a_4 - 21a_2^2a_3 + 3a_3^2 + 14a_2^4. \end{aligned} \tag{1.3}$$

Using the definition of the second Hankel determinant, together with (1.2) and (1.3), after some calculations we have

$$\begin{aligned} H_2(2)(f^{-1}) = A_2A_4 - A_3^2 &= a_2a_4 - a_3^2 - a_2^2(a_3 - a_2^2) \\ &= H_2(2)(f) - a_2^2(a_3 - a_2^2). \end{aligned} \tag{1.4}$$

For our consideration we need the next lemma proven in [15] (can be found also in [1]).

**Lemma 1.1.** *Let*

$$\omega(z) = c_1 z + c_2 z^2 + \cdots \tag{1.5}$$

*be a Schwartz function. Then*

$$|c_1| \le 1, \quad |c_2| \le 1 - |c_1|^2 \quad \textit{and} \quad |c_3| \le 1 - |c_1|^2 - \frac{|c_2|^2}{1 + |c_1|}.$$

## 2. Main results

In the main theorem we will give upper bound (mainly sharp) of the modulus of the second Hankel determinant for the inverse functions of functions from some classes univalent functions. The same problem was studied in [13] for the class $\mathcal{U}(\lambda)$, $0 < \lambda \le 1$, of univalent functions defined by

$$\mathcal{U}(\lambda) = \left\{ f \in \mathcal{A} : \left| \left( \frac{z}{f(z)} \right)^2 f'(z) - 1 \right| < \lambda, z \in \mathbb{D} \right\}.$$

**Theorem 2.1.** *The next statements are valid.*

$(i)$ *If $f \in \mathcal{C}$, then $|H_2(2)(f^{-1})| \le \frac{1}{8}$.*
$(ii)$ *If $f \in \mathcal{S}^*$, then $|H_2(2)(f^{-1})| \le 3$.*
$(iii)$ *If $f \in \mathcal{G}$, then $|H_2(2)(f^{-1})| \le \frac{1}{16}$.*
$(iv)$ *If $f \in \mathcal{S}^*_q$, then $|H_2(2)(f^{-1})| \le \frac{11}{38}$.*

*The first three inequalities are sharp.*

*Proof.*

$(i)$ From the definition of the class $\mathcal{C}$, for $f \in \mathcal{C}$, we have:

$$1 + \frac{zf''(z)}{f'(z)} = \frac{1+\omega(z)}{1-\omega(z)}, \tag{2.1}$$

where $\omega$ is a Schwartz function. From (2.1) we have

$$(zf'(z))' = [1 + 2\omega(z) + 2\omega^2(z) + \cdots]f'(z),$$

and from here by using the notations (1.1) and (1.5) and equating the coefficients:

$$\begin{aligned} a_2 &= c_1, \\ a_3 &= \tfrac{1}{3}(c_2 + 3c_1^2), \\ a_4 &= \tfrac{1}{6}(c_3 + 5c_1c_2 + 6c_1^3). \end{aligned} \tag{2.2}$$

Using (1.4) and (2.2), after some calculations, we obtain

$$H_2(2)(f^{-1}) = \frac{1}{6}c_1c_3 - \frac{1}{6}c_1^2c_2 - \frac{1}{9}c_2^2, \tag{2.3}$$

which implies

$$|H_2(2)(f^{-1})| \le \frac{1}{6}|c_1||c_3| + \frac{1}{6}|c_1|^2|c_2| + \frac{1}{9}|c_2|^2.$$

If we use Lemma 1.1 for $|c_3|$, then from the last relation we have

$$\begin{aligned} |H_2(2)(f^{-1})| &\le \frac{1}{6}|c_1|\left(1 - |c_1|^2 - \frac{|c_2|^2}{1+|c_1|}\right) + \frac{1}{6}|c_1|^2|c_2| + \frac{1}{9}|c_2|^2 \\ &= \frac{1}{6}|c_1|(1-|c_1|^2) + \frac{2-|c_1|}{18(1+|c_1|)}|c_2|^2 + \frac{1}{6}|c_1|^2|c_2|, \end{aligned}$$

and if we use that $|c_2| \le 1 - |c_1|^2$ (again from Lemma 1.1) and some calculations:

$$|H_2(2)(f^{-1})| \leq \frac{1}{9}(1+|c_1|^2-2|c_1|^4) \leq \frac{1}{8},$$

where maximum is attained for $|c_1| = \frac{1}{2}$. This result is the best possible. Namely, if in (2.1) we choose the Schwartz function

$$\omega_1(z) = z\frac{z+\frac{1}{2}}{1+\frac{1}{2}z} = \frac{1}{2}z + \frac{3}{4}z^2 - \frac{3}{8}z^3 + \cdots,$$

with $c_1 = \frac{1}{2}$, $c_2 = \frac{3}{4}$, $c_3 = -\frac{3}{8}$, then we receive the function $f_1$ defined by

$$1 + \frac{zf_1''(z)}{f_1'(z)} = \frac{1+\omega_1(z)}{1-\omega_1(z)},$$

such that, from (2.3), $H_2(2)(f_1^{-1}) = -\frac{1}{8}$, i.e., $|H_2(2)(f_1^{-1})| = \frac{1}{8}$.

$(ii)$ We will use the same method as in the case $(i)$. If $f \in \mathcal{S}^*$, then by definition

(2.4)
$$\frac{zf'(z)}{f(z)} = \frac{1+\omega(z)}{1-\omega(z)}$$

for some Schwartz function $\omega$. First, from the relation (2.4) we have

$$zf'(z) = [1 + 2\omega(z) + 2\omega^2(z) + \cdots] \cdot f(z),$$

and using the notations (1.1) and (1.5), after comparing the coefficients:

(2.5)
$$\begin{aligned} a_2 &= 2c_1, \\ a_3 &= c_2 + 3c_1^2, \\ a_4 &= \tfrac{2}{3}(c_3 + 5c_1c_2 + 6c_1^3). \end{aligned}$$

Using (1.4) and (2.5), after some calculations we obtain

$$H_2(2)(f^{-1}) = \frac{4}{3}c_1c_3 - \frac{10}{3}c_1^2c_2 - c_2^2 + 3c_1^4,$$

and from here

$$|H_2(2)(f^{-1})| \leq \frac{4}{3}|c_1||c_3| + \frac{10}{3}|c_1|^2|c_2| + |c_2|^2 + 3|c_1|^4.$$

Further, using Lemma 1.1 and the same estimation as in the previous case, we have

$$|H_2(2)(f^{-1})| \leq \frac{1}{3}(3+8|c_1|^2-2|c_1|^4) \leq 3,$$

where the maximum is attained when $|c_1| = 1$. For the Koebe function $k(z) = \frac{z}{(1-z)^2}$ we have $c_1 = 1$ and $c_i = 0$ for $i = 2, 3, \ldots$, i.e., $H_2(2)(k^{-1}) = 3$, which means that our result is sharp.

$(iii)$ From the definition of the class $\mathcal{G}$, for $f \in \mathcal{G}$, we have

$$1 + \frac{zf''(z)}{f'(z)} = \frac{3}{2} - \frac{1}{2} \cdot \frac{1+\omega(z)}{1-\omega(z)} \tag{2.6}$$

for some Schwartz function $\omega$. From (2.6) we have

$$(zf'(z))' = [1 - \omega(z) - \omega^2(z) - \cdots] \cdot f'(z),$$

and from here, by using (1.1) and (1.5) and equating the coefficients:

$$\begin{aligned} a_2 &= -\tfrac{1}{2}c_1, \\ a_3 &= -\tfrac{1}{6}c_2, \\ a_4 &= -\tfrac{1}{24}(2c_3 + c_1c_2). \end{aligned} \tag{2.7}$$

Using (1.4) and (2.6), after some calculations, we get

$$H_2(2)(f^{-1}) = \frac{1}{144}(6c_1c_3 + 9c_1^2c_2 + 9c_1^4 - 4c_2^2) \tag{2.8}$$

and from here

$$|H_2(2)(f^{-1})| \le \frac{1}{144}(6|c_1||c_3| + 9|c_1|^2|c_2| + 9|c_1|^4 + 4|c_2|^2).$$

Using the estimations given in Lemma 1.1 and the same method as in two previous cases, we have

$$|H_2(2)(f^{-1})| \le \frac{1}{144}(4 + 7|c_1|^2 - 2|c_1|^4) \le \frac{1}{16},$$

where maximum is attained for $|c_1| = 1$. For the function $f_2$ defined by

$$1 + \frac{zf_2''(z)}{f_2'(z)} = \frac{3}{2} - \frac{1}{2} \cdot \frac{1+z}{1-z}$$

(i.e. when $\omega(z) = z$ in (2.6)) we have $c_1 = 1$, $c_2 = c_3 = 0$, which, by (2.8), implies $H_2(2)(f_2^{-1}) = \frac{1}{16}$, i.e., the result is sharp.

$(iv)$ From the definition of the class $\mathcal{S}_q^*$ for $f \in \mathcal{S}_q^*$, we have

$$\frac{zf'(z)}{f(z)} = \omega(z) + \sqrt{1 + \omega^2(z)} \tag{2.9}$$

for some Schwartz function $\omega$. From the relation (2.4) we have

$$zf'(z) = \left[\omega(z) + \sqrt{1+\omega^2(z)}\right] \cdot f(z).$$

Using the notations (1.1) and (1.5), and after comparing the coefficients we receive

$$\begin{aligned} a_2 &= c_1, \\ a_3 &= \tfrac{1}{2}c_2 + \tfrac{3}{4}c_1^2, \\ a_4 &= \tfrac{1}{12}(4c_3 + 10c_1c_2 + 5c_1^3). \end{aligned} \tag{2.10}$$

Using (1.4) and (2.10), after some calculations we obtain that

$$H_2(2)(f^{-1}) = \frac{1}{3}c_1c_3 - \frac{5}{12}c_1^2c_2 + \frac{5}{48}c_1^4 - \frac{1}{4}c_2^2,$$

and from here

$$|H_2(2)(f^{-1})| \leq \frac{1}{3}|c_1||c_3| + \frac{5}{12}|c_1|^2|c_2| + \frac{5}{48}|c_1|^4 + \frac{1}{4}|c_2|^2.$$

If we use, once again, Lemma 1 and the same method as in three previous cases we obtain

$$|H_2(2)(f^{-1})| \leq \frac{1}{48}(-19|c_1|^4 + 12|c_1|^2 + 12),$$

where maximum $\frac{11}{38}$ is attained for $|c_1| = \sqrt{\frac{6}{19}} = 0.56195\ldots.$

□

The expression (1.4) gives

$$H_2(2)(f^{-1}) - H_2(2)(f) = -a_2^2(a_3 - a_2^2), \tag{2.11}$$

which together with the estimates from the previous theorem easily gives the following corollary.

**Corollary 2.1.** *The next statements are valid.*

$(i)$ *If $f \in \mathcal{C}$, then $|H_2(2)(f^{-1}) - H_2(2)(f)| \leq \frac{1}{12}$.*
$(ii)$ *If $f \in \mathcal{S}^*$, then $|H_2(2)(f^{-1}) - H_2(2)(f)| \leq 4$.*
$(iii)$ *If $f \in \mathcal{G}$, then $|H_2(2)(f^{-1}) - H_2(2)(f)| \leq \frac{1}{16}$.*
$(iv)$ *If $f \in \mathcal{S}_q^*$, then $|H_2(2)(f^{-1}) - H_2(2)(f)| \leq \frac{1}{4}$.*

*All inequalities are sharp.*

*Proof.*

$(i)$ From the relations (2.11) and (2.2), in this case we have

$$|H_2(2)(f^{-1}) - H_2(2)(f)| = |a_2|^2|a_3 - a_2^2| = \frac{1}{3}|c_1|^2|c_2|$$
$$\le \frac{1}{3}|c_1|^2(1-|c_1|^2) \le \frac{1}{12},$$

attained for $|c_1|^2 = \frac{1}{2}$. This estimate is sharp, because as in case of the function $f_1$, we can choose now the Schwartz function

$$\omega_2(z) = z\frac{z+c}{1+cz} = cz + (1-c^2)z^2 - c(1-c^2)z^3 + \cdots,$$

with $c_1 = c = \frac{1}{\sqrt{2}}$ and $c_2 = 1 - c_1^2 = \frac{1}{2}$, leading to the function $f_3$, such that

$$H_2(2)(f_3^{-1}) - H_2(2)(f_3) = -a_2^2(a_3 - a_2^2) = -\frac{1}{3}c_1^2c_2 = -\frac{1}{12}.$$

$(ii)$ From the relations (2.11) and (2.5), we have

$$|H_2(2)(f^{-1}) - H_2(2)(f)| = 4|c_1|^2|c_2 - c_1^2| \le 4|c_1|^2 \le 4,$$

with equality for the Koebe function ($c_1 = 1$, $c_2 = 0$).

$(iii)$ From (2.11) and (2.7) we receive

$$\begin{aligned}|H_2(2)(f^{-1}) - H_2(2)(f)| &= \frac{1}{4}|c_1|^2 \cdot \left|-\frac{1}{6}c_2 - \frac{1}{4}c_1^2\right| \\ &\le \frac{1}{4}|c_1|^2 \cdot \left(\frac{1}{6}|c_2| + \frac{1}{4}|c_1|^2\right) \\ &\le \frac{1}{48}|c_1|^2 \cdot (2 + |c_1|^2) \\ &\le \frac{1}{16},\end{aligned} \tag{2.12}$$

where we used $|c_2| \le 1 - |c_1|^2$, and where the equality sign is attained for the function $f$ defined be (2.6) and $\omega(z) = z$.

$(iv)$ Using the same method as in three previous remarks, we easily obtain that for the class $\mathcal{S}_q^*$:

$$\begin{aligned}|H_2(2)(f^{-1}) - H_2(2)(f)| &= \frac{1}{2}\cdot|c_1|^2\cdot\left|c_2-\frac{1}{2}c_1^2\right| \\ &\le \frac{1}{2}\cdot|c_1|^2\cdot\left(|c_2|+\frac{1}{2}|c_1|^2\right) \\ &\le \frac{1}{2}\cdot|c_1|^2\cdot\left(1-\frac{1}{2}|c_1|^2\right) \\ &\le \frac{1}{4},\end{aligned}$$

This estimation is sharp as the function $f_4$ defined by

$$\frac{zf_4'(z)}{f_4(z)} = z+\sqrt{1+z^2},$$

i.e., for $\omega(z)=z$ in (2.9).

□

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