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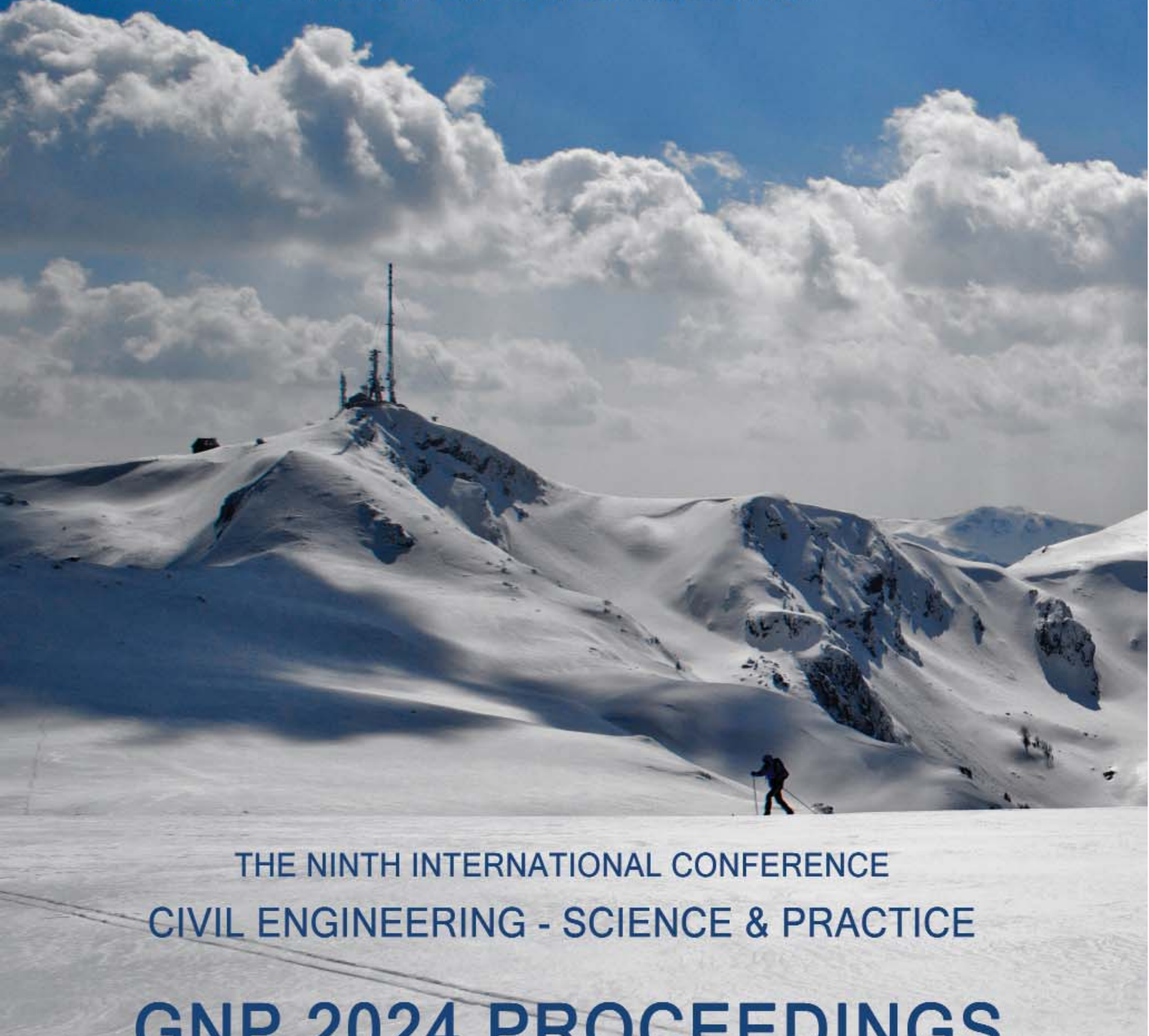

THE NINTH INTERNATIONAL CONFERENCE

CIVIL ENGINEERING - SCIENCE & PRACTICE

GNP 2024 PROCEEDINGS

Kolašin, March 2024

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ANALYTIC AND EXPERIMENTAL DETERMINATION IMPULSE RESPONSE OF SINGLE DEGREE OF FREEDOM SYSTEM

Abstract

The paper presents an experimental and analytical way of determining the impulse response of a linear time-invariant damped system with one degree of freedom.

An experiment was conducted where the mass was excited by an impact load. Both the mass acceleration and impact force values were recorded. The magnitude of the system's frequency response was determined based on the recorded values, ensuring the relatively simple identification of the system's basic parameters. Those parameters were used to define a suitable mathematical model of the system, transfer function, and frequency response function in an analytical form.

The impulse response of the system based on the recorded values of mass acceleration and impact load is expressed as a discrete function. This function is determined by applying the inverse discrete Fourier transform of the corresponding frequency response function.

The impulse response of the system, based on the mathematical model, is expressed as a continuous function. This function is determined by applying the inverse Fourier transform of the corresponding frequency response.

Finally, it was shown that the response of a system to arbitrary load could be determined by convolving the impulse response of the system with the load function. Convolution of continuous functions is difficult to perform, and it can be used only for the simplest problem and for understanding the physical phenomenon. Discrete convolution has a practical utility because it is easy to perform in some program languages like Matlab. However, discrete impulse response has limitations depending on the test conditions.

Keywords

Transfer function, acceleration, single degrees of freedom system, system response

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1. INTRODUCTION

The excitation response of a dynamic system can be calculated using the finite element method. This method is usually effective for a selected numerical model of a real structure or element. The problem is that real structures are usually complex and cannot be described simply by numerical or analytical procedures. Those models can be improved with experimentally determined parameters. In other words, model updating can be used to update parameters such as natural frequencies, damping ratios, mode shapes, etc. In this way, the structure's response under load can be better predicted according to the selected numerical or analytical model. Structure response can be determined entirely based on experiment. In that case, the structure test implies to record input–applied load and output–structure response. Based on that, the structure's impulse response can be determined. This response includes every significant property of the structure, including the abovementioned parameters. The benefit of knowing the impulse response function (IRF) is that it can calculated the response on any load function using only the convolution of IRF and that load function.

Real structures have an infinite number of degrees of freedom. Therefore, they have an infinite number of impulse responses. A real system with a continuously distributed mass can be discretized in a finite number of mass points. Consequently, loads that act in one point excite response in all these points. Thus, the multi-input-multi output (MIMO) system can be formed. Impulse response $\mathrm{h}_{\mathrm{m,n}}$ represents the response at place m due to the act of ideal impulse force at place n, where $\mathrm{h}_{\mathrm{m,n}} = \mathrm{h}_{\mathrm{n,m}}$, and , $\mathrm{m}, \mathrm{n} = 1,2, .. \mathrm{M}$, where M is a finite number of discretized mass.

For the sake of simplicity, without loss of generality, this paper presents determining impulse response for linear time-invariant damping single degree of freedom (single input-single output, SISO system). An experiment-based method for determining impulse response was demonstrated using a mathematical model to validate the impulse response functions.

Impulse response and following functions based on the corresponding mathematical model are continuous (analog) functions. On the other hand, the system's impulse response and the following functions based on the conducted experiment are expressed in discrete form (digital form of functions).

2. A SHORT THEORY REVIEW OF SDOF SYSTEMS

A linear, time-invariant (LTI) model with viscous damping is considered. The differential equation of motion can be represented in the following form:

$$\ddot{\mathrm{x}}(\mathrm{t}) + 2\epsilon\,\dot{\mathrm{x}}(\mathrm{t}) + \omega_s^2\,\mathrm{x}(\mathrm{t}) = \frac{\delta(\mathrm{t})}{m};\quad \forall t \in (0^-, +\infty) \tag{1}$$

where $\mathrm{x}(\mathrm{t})$ is displacement, t is continuous time variable, ω_s is the system's circular frequency, viscous damping coefficient can be calculated as $\epsilon = \zeta\,\omega_s$, ζ is the damping ratio, $\delta(\mathrm{t})$ is the Dirac *delta* function and m is the mass of system. For all $t \in (-\infty, 0^-)$ system response and loads is equal to zero (causal system and functions). It is also assumed that all initial conditions in (1) are equal to zero. Therefore, equation (1) can be expressed in the complex "s" plane using the Laplace transform as follows:

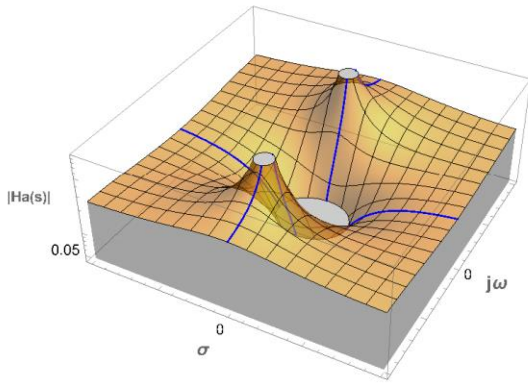

Figure 1. Magnitude of acceleration transfer function

$$\mathrm{X(s)} = \mathrm{Hx(s)P(s)} = \frac{1}{m}\frac{1}{(s-\varepsilon)^2+\omega_d^2} = \mathrm{Hx(s)} \tag{2}$$

where the damped circular frequency of the system is determined as $\omega_d = \omega_s\sqrt{1-\zeta^2}$ and variable "s" as $\mathrm{s} = \sigma + \mathrm{j}\omega$. Real part of variable "s" is σ, ω is circular frequency and $j = \sqrt{-1}$. System response in "s" plane X(s) is defined as product of transfer function Hx(s) and Laplace transform of applied load P(s). Response of the system X(s) is actually transfer function Hx(s) (displacement transfer function) because the Laplace transform of Dirac *delta* function is equal to 1 [4, 5]. The Acceleration transfer function is obtained multiplying Hx(s) by "s^2" [4, 6].

$$\mathrm{Ha(s)} = \frac{1}{m}\frac{s^2}{(s-\varepsilon)^2+\omega_d^2} \tag{3}$$

The magnitude of equation (3) for arbitrary parameters is shown in Figure 1. The roots of the numerator in expression (3) are poles of the transfer function or damped natural frequencies.

Impulse response of the system is inverse Laplace transform of Ha(s):

$$\mathrm{ha(t)} = \frac{1}{m}\delta(t) + \frac{1}{m\omega_d}e^{-\epsilon t}\left((\epsilon^2 - \omega_d^2)\sin(\omega_d t) - 2\epsilon\omega_d\cos(\omega_d t)\right);\ \forall t \in (0^-, +\infty) \tag{4}$$

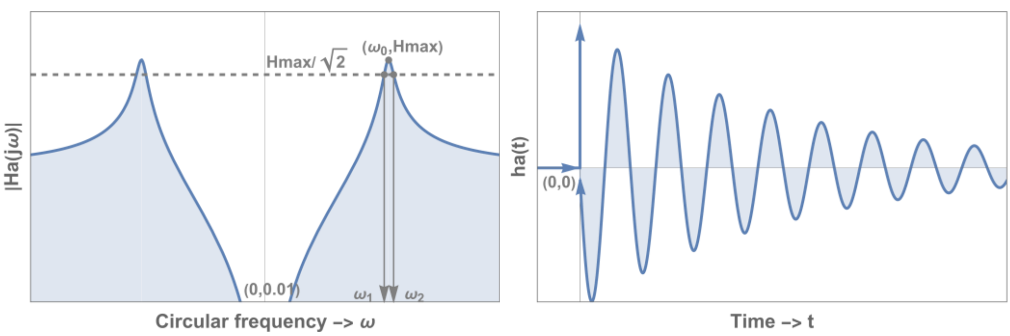

Figure 2. Magnitude of Frequency Response – Accelerance - left, Acceleration Impulse Response - right

The function (4) tends to $-2\epsilon/\mathrm{m}$ in the point $\mathrm{t}=0^{+}$ and in the point $\mathrm{t}=0$ it contains Dirac *delta* function, see Figure 2 right. By replacing "s" with "jω" the equation (3) becomes frequency response function (FRF) or accelerance FRF:

$$\mathrm{Ha}(\mathrm{j}\omega)=\frac{1}{m}\frac{-\omega^{2}}{(j\omega-\varepsilon)^{2}+\omega_{d}^{2}} \tag{5}$$

Magnitude of FRF (accelerance FRF) is shown on the Figure 2 left. The parameter $\omega_0 \approx \omega_s$ is determined by simply picking the maximum point on the magnitude of FRF plot (accelerance plot). This is adequate for small value of ζ ($\zeta \ll 1$) becouse the system frequency can be calculated as $\omega_s=\omega_0\sqrt{1-2\zeta^2}$. The damping ratio ζ can be calculated with "3dB" method (Half power method) as depicted on Figure 2 left. Negative frequencies on Figure 2 left are not of interest. Such a way of determining system parameters ω_s and ζ can be used both for continuous and discrete functions.

3. EXPERIMENT DESCRIPTION

The model contains a mass $\mathrm{m}=4.165$ [kg]. Mass is suspended in such a way to allow motion in the direction of applied impact load and limit parasitic rotational and transfersal motion components.

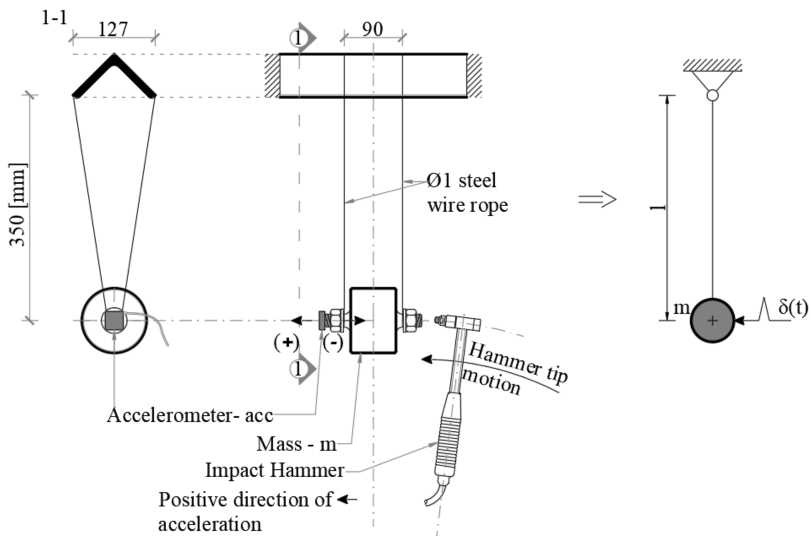

Figure 3. Eexperiment description

Acceleration response was regstred with an accelerometer with sensitivity 2000 [mV/g] ($\mathrm{g}=9.81[\mathrm{m/s^2}]$) and frequency range up to ≈ 300 [Hz] [7] in the direction of applying the force. Force was applied with impact hammer [8] with sensitivity 2.25 [mV/N] and force range ± 2224 [N]. The data acquisition was performed by the 24-bit universal measuring amplifier [9]. During the data acquisition process, the sampling rate $fs=9600$ $[Hz]$ was adopted. Figure 3 ilustrates the conducted experiment and corresponding mathematical model.

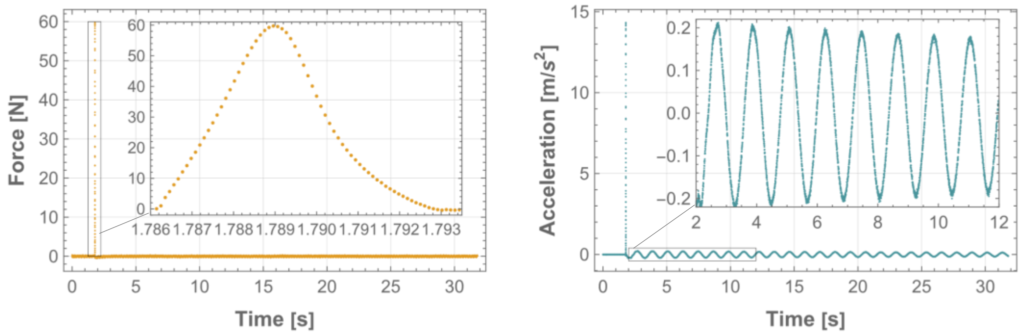

Figure 4. Recorded force - left and acceleration - right during measurements

Figure 4 shows the registered records of impact force and acceleration. Figure 4 on the left shows the recorded impact force and the part corresponding only to the time during the impact for clarity. In the same Figure right, the acceleration record is shown with an enlarged part from the second to the 12th second for a better display of the mass oscillations after the impact. The shape of the recorded acceleration during the impact corresponds to the shape of the recorded impact force.

For the data analysing procedure, a part of the acceleration record from the moment just before the impact until the end of the record was adopted. The impact force for analysis corresponds to part of the record during the impact. The rest of the samples of the record are replaced by zeros because it contain noise and registrated a parasitic low level of inertia force as a result of the movement of the operator arm after the impact.

4. EXPERIMENTAL AND ANALYTIC RESULTS

During the measurement, records of acceleration acc[n] and impact force p[n] were registered, where $n = 0,1,2,\dots,N-1$. N is the total number of samples. A discrete Fourier transform of acc[n] and p[n] are the follwing functions, respectively:

$$\mathrm{Acc}[k] = \sum_{n=0}^{N-1} \mathrm{acc}[n] e^{-\frac{j2\pi}{N}kn}\,; \mathrm{P}[k] = \sum_{n=0}^{N-1} \mathrm{p}[n] e^{-\frac{j2\pi}{N}kn}\,;\ k, n = 0,1,2, \dots N-1 \tag{6}$$

Figure 5 left shows the magnitude of Fourier transforms of registered values based on equations (6). It can be noted that there is a significant level of signal to noise ratio for frequencies above ≈ 300 [Hz] for the acceleration record, which is expected based on the metrological characteristics of the accelerometer. The impact force is not an ideal impulse. It has a finite duration, so the FRF (Accelerance) of the system is determined as follows:

$$\mathrm{FRF}[k] = \frac{\mathrm{Acc}[k]}{\mathrm{P}[k]};\ k = 0,1,2, \dots N-1 \tag{7}$$

The magnitude of the equation (7) is shown on Figure 5 right. By dividing recorded values, the noise increases for higher frequencies than ≈ 300 [Hz]. That part needs to be removed. An elliptic (IIR – infinite impulse response) low-pass filter of the seventh order is adopted. Magnitude filter characteristic specifications are: minimal stopband attenuation $\delta s = 60$ [dB], cut of frequency – transition zone spans from $fp = 300$ to $fst = 400$ [Hz], passband riple tolerance $\delta p = 0.3$ [dB]. The Matlab function "*elipord*" was applied to determine the required filter order to meet the specified requirements. The coefficients of the filter are determined by the Matlab function "ellipse".

The filtered frequency response $\mathrm{FRF_{filt}}[\mathrm{k}], \mathrm{k} = 0,1,2, \dots \mathrm{N}-1$ was calculated as Fourier Transform of filtered impulse response $\mathrm{ha}_{filt}[n], \mathrm{n} = 0,1,2, \dots \mathrm{N}-1$. Function (vector) $\mathrm{ha}_{filt}[n]$ was determined with Matlab functions "*filter*" using vector of inverse Fourier transform of FRF[k] and filter corfficients. Figure 5 right shows the magnitude of the filtered frequency response $\mathrm{FRF}_{filt}[\mathrm{k}]$.

The system parameters ω_s and ζ can be determined by peack-picking method and "3dB" method [2], as mentioned at the end of the previous section, using magnitude of FRF[k] or $\mathrm{FRF}_{filt}[\mathrm{k}]$ fuction. However, frequency resolution is not inaf high for adequate damping identification. The duration of records should have been longer. Those parameters can be aproximately determined with optimal choise paremeters ω_d, ε and V to fit continuous function $\overline{acc}(\mathrm{t})$ based on equation (4) with recorded data acc[n] (from end of impact to end record). Equation (8) is aproximately response to applied impact force p[n]. Those parameters are calculated using function "*NonlinearModelFit*" in Mathematica program languege.

$$\overline{\mathrm{acc}}(\mathrm{t}) = \frac{\mathrm{V}}{\mathrm{m}\omega_\mathrm{d}} \mathrm{e}^{-\epsilon\mathrm{t}} \left((\epsilon^2 - \omega_\mathrm{d}^2) \sin(\omega_\mathrm{d}\mathrm{t}) - 2\epsilon\omega_\mathrm{d} \cos(\omega_\mathrm{d}\mathrm{t}) \right) \tag{8}$$

The required parameters are $\omega_s \approx 5.26\,[rad/s], \epsilon \approx 0.02$ and $V \approx 0.17$. When the parameters ω_s and ζ are replaced in equation (5), an analytical expression (exact solution) for the magnitude FRF characteristic can be defined as a continuous function in Figure 5 right. A suitable match can be seen in the frequency range $\approx (0.4 - 300)\ [Hz]$ when comparing the continuous and discrete filtered magnitude characteristics.

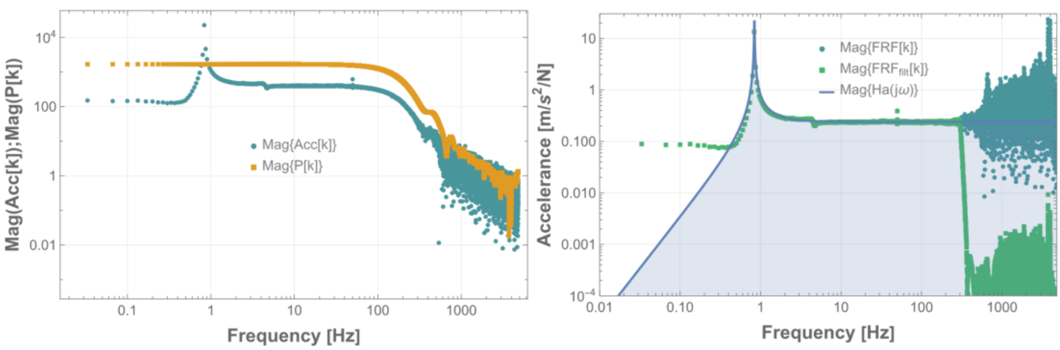

Figure 5. Magnitude specter of recorded signals- left and Accelerance FRF- right

The function $\mathrm{ha}_{filt}[\mathrm{n}]$ can be compared with function ha(t) when the experimentally determined parameters ω_s and ζ are replaced in equation (4), as shown in Figure 6.

Figure 6 left illustrates the first 4 seconds of impulse responses for clarity. The total duration of $\mathrm{ha}_{filt}[n]$ equals to $\mathrm{N} \cdot \mathrm{T} \approx 30$ [s]. On the other hand, exact impulse response ha(t) has infinite duration ($t \in (0^-, +\infty)$). At the same picture on the right, only the first 0.08 [s] is shown to compare the part corresponding to the impact determined experimentally with the exact.

Now, when the experimentally impulse response is known, response to arbitrary excitation – load can be determined. To verify that, $\mathrm{ha}_{filt}[\mathrm{n}]$ can be used to determine response to applied load p[n]. This response $\widehat{\mathrm{acc}}[\mathrm{n}]$ should corespond to measure response acc[n] and it can be calculated as convolution of vectors p[n] and $\mathrm{ha}_{filt}[n]$ as follows:

$$\widehat{\mathrm{acc}}[\mathrm{n}] = \mathrm{T}_s \sum_{\mathrm{k}=0}^{\mathrm{n}} \mathrm{p}[\mathrm{k}]\mathrm{ha}_{filt}[n-k]\,;\ n = 0,1,2, \dots, N-1 \tag{9}$$

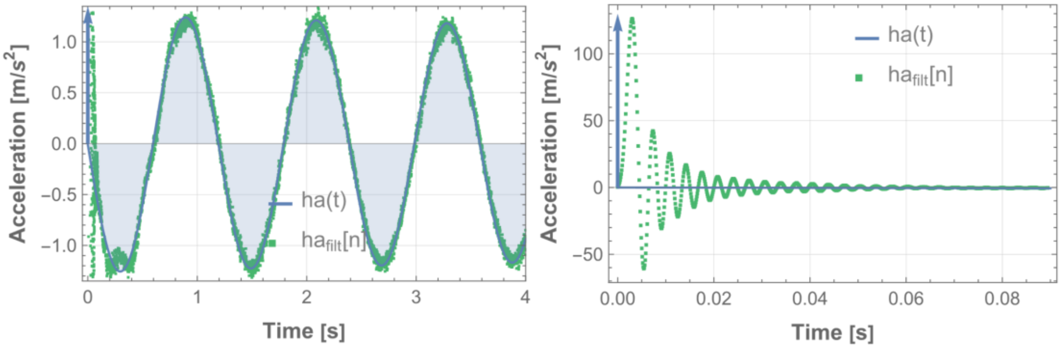

Figure 6. Analytic and numeric system impulse response

where the sampling period is calculated as $T_s = 1/f_s$.

Figure 7 shows plots of recorded response acc[n] and calculated response $\widehat{acc}[n]$. At the Figure 7 right is shown only four seconds of response for clarity. It is evident that those functions are well correlated. There are delay of 20-40 samples. It is due to filter delay characteristic. Noise that exist in vector (function) $\widehat{acc}[n]$ could be lower level if the cut of frequency of filter was smaller than chosen 300 [Hz]. However, riples after the main lobe at the biginning $\widehat{acc}[n]$ due to the impact are bigger as the cut of frequency decreases. At the same thime the maximum acceleration in the pick become smaller. Becouse of the that, the cut of frequency would not be smaller then $10f_{sis,max}$, where $f_{sis,max}$ is natural frequency (SISO system) or the highest natural frequencies of interest for multi degree of freedom systems. In the other words, higher natural modes of vibration than chosen in such a way in the response would not be properly reconstruct. In the other hand $ha_{filt}[n]$ would not be used for load with smaller frequency of ≈ 0.4 [Hz], see Figure 5 right. In this region magnitude of FRF_{filt} function deviates significantly from exact function ha(t).

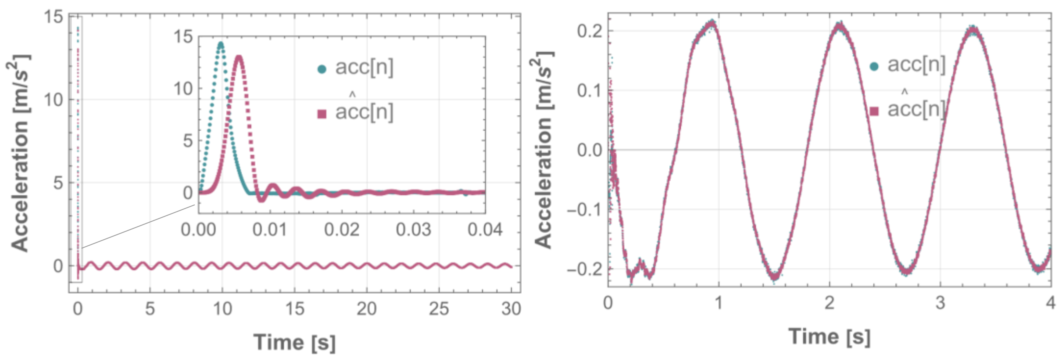

Figure 7.Experiment acceleration record and reconstructed response

5. CONCLUSION

The method of determining the system's impulse response by an experimental procedure in the discrete form is presented and compared with the solution in a closed mathematical form. An LTI system with single degree of freedom was chosen so that the mathematical model could be as simple as possible and, on the other hand, similar to the physical-real model. The same procedure can be applied to a system with multi degrees of freedom. It is necessary to repeat the procedure of applying the known force while recording the response at desired location.

The impulse response is useful function because it can be used to predict response to arbitrary load (excitation) by convolution of those functions. Convolution of continuous functions is difficult to perform, and it can be used only for the simplest problem and for understanding the physical phenomenon. Discrete convolution has a practical utility because it is easy to perform. However, there are some limitations, such as the impulse response of civil structures is always infinite but it can be register a finite number of samples. Likewise, applied devices for quantity measurements have their own metrological limitations properties. In addition, there is a problem with applying the impact load on real structures and its simultaneous recording.

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