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On the Properties of the Concave Antiprisms of Second Sort

The paper examines geometrical, static and dynamic properties of the polyhedral structures obtained by folding and creasing the two-rowed segment of equilateral triangular net. Bases of these concave polyhedra are regular, identical polygons in parallel planes, connected by the alternating series of triangles, as in the case of convex antiprisms. There are two ways of folding such a net, and therefore the two types of concave antiprisms of second sort. The paper discusses the methods of obtaining the accurate position of the vertices and other linear parameters of these polyhedra, with the use of mathematical algorithm. Structural analysis of a representative of these polyhedra is presented using the SolidWorks program applications.

Keywords: *polyhedron, concave, equilateral triangle, hexahedral element.*

1. INTRODUCTION

Concave antiprism of the second sort is a polyhedron consisting of two regular polygonal bases in parallel planes and deltahedral lateral surface. The lateral surface consists of two-row strip of equilateral triangles, arranged so to form spatial hexahedral elements, by whose polar arrangement around the axis that connects bases' centroids, the polyhedron is created. The process of generating these polyhedra is similar to the formation of concave cupolae of second sort [6], [7], as well as polygrammatic antiprisms [1], [2] or pseudo-cylinders [4]. It is also akin to origami technique [6], since the folding of planar net by the assigned edges produces 3D structures.

In this paper, we deal with the problem of defining geometric properties of these polyhedral structures in order to analyze their behavior on static and dynamic effects, which will show whether these structures are suitable for applications in engineering. In this sense, we should firstly define accurate vertex positions for the observed base, i.e. for the assigned number n of its sides.

2. FORMATION OF THE CONCAVE ANTIPRISMS OF SECOND SORT

The lateral surface net of the concave antiprism of second sort (CA II, also in the further text) is a segment of an equilateral triangular plane tessellation, as shown in Fig. 1.a and Fig. 1.b.

By folding and joining the corresponding edges of the net, the closed ring is obtained, which is a fragment of a polyhedral surface. Number of the unit cells – spatial hexahedral elements – is determined by the number of the base sides. Knowing that it is impossible to form a convex polyhedral vertex figure with the six equilateral triangles arranged around the common vertex, we conclude that the formed fragment of polyhedral surface must be concave.

There are two ways of folding the net, with the internal and with the external vertex G of the spatial hexahedral element, depending of which there will

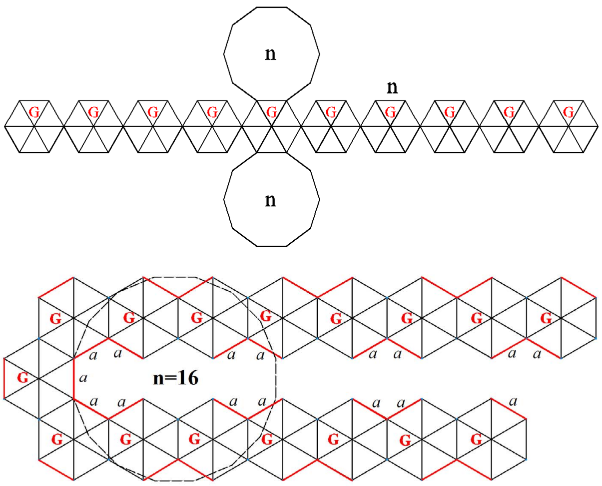

Figure 1. The two methods of forming plane net of the concave antiprism of second sort. a) Linear two-row strip of hexagonal elements joined by the vertices. b) Continual hexahedral elements joined by the sides

appear two types of each concave antiprism of second sort which differ in their heights.

Hence, with the same polygonal base, there will appear two variations of concave antiprism of second sort: a) CA II-M having the internal vertex G of the spatial hexahedral element ABCDEFG, with the major height, b) CA II-m having the external vertex G of the spatial hexahedral element ABCDEFG, with the minor height (as shown in Fig. 2 a and b), which is similar to the formation of concave cupola of the second sort [1].

The basis which form a concave antiprism of second sort can be any regular polygon, starting from n=3 (n=5 for the variant "m"), and ending with n=∞, when the series of linearly arranged spatial hexahedral elements will be formed.

Note: If we comply with the terms of eliminating cases with self-intersecting faces, for the variant CA II-m, the

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basis from which it is possible to form the concave antiprism of second sort will be n=5, because n=3 leads to self intersecting faces, while for n=4, the solid we obtain will be known non-convex uniform polyhedron Octahemioctahedron [6].

In Fig. 2, we show the examples of the two variations of forming the lateral surface of the concave antiprisms with hexadecagonal basis in two orthogonal projections: the top view and the front view.

In Fig. 3-a and Fig 3-b, the axonometric view on the same lateral surfaces' rings are shown.

Unlike some previous researches on the similar topic [5,7-11] these polyhedra are not pseudo cylindrical, because the vertex disposition of the concave antiprisms of second sort does not follow the cylindrical form, i.e. the cylinder could not be circumscribed around the vertices of these structures, but rather the spherical/elliptical rings, the segments of elliptical or toroidal surface (Fig 3-c).

In Fig. 4 (a and b) we show only one fragment of each type of concave antiprism of second sort's net, a spatial hexahedral element ABCDEFG. It consists of six equilateral triangles formed around a common vertex G.

Prior to defining the parameters of these solids, it is necessary to set the initial conditions for this spatial hexahedron, which it must meet in order to form a closed geometric unit - the lateral surface of the concave antiprism of second sort - by radial arranging the identical cells around the axis k:

- α is a vertical symmetric plane of the polygonal sides (polyhedral edges) AB and DE, whereat the vertex G belongs to the plane α,
- Vertical plane β is determined by the axis k and the vertices B, C and D,
- The edges AB and DE are horizontal, and the vertices A and E, as well as B and D are in the same vertical planes, respectively,
- The edges CG and FG belong to the horizontal plane γ, which is set at the half height of the spatial hexahedral element ABCDEFG

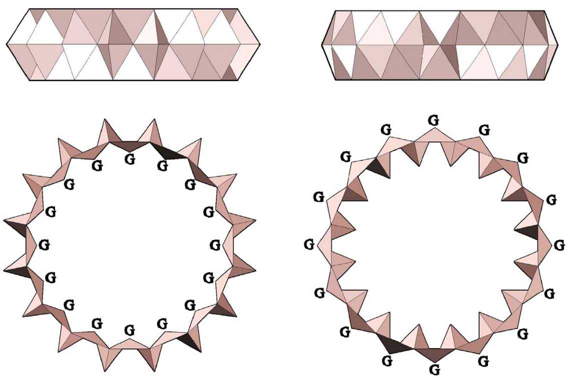


Figure 2. Ther lateral surface of the concave antiprism of second sort: a) with the internal vertex G of the spatial hexahedral element ABCDEFG, b) with the external vertex G of the spatial hexahedral element ABCDEFG

In order to determine the parameters of this structure, we can use:

a) Methods of descriptive geometry
b) iterative algebraic methods
c) using the mechanical model.

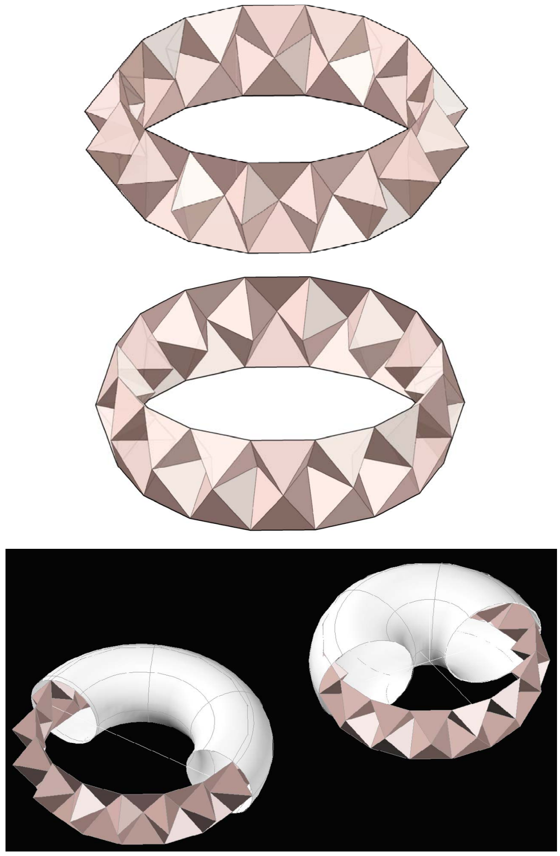
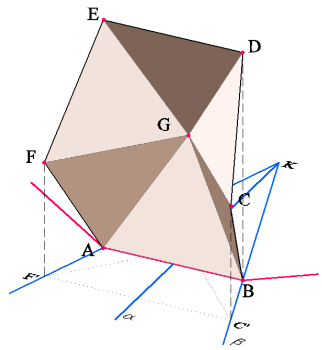
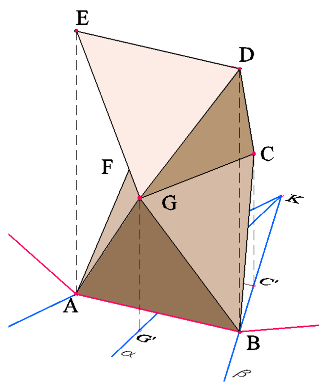


Figure 3. axonometric view of the two variations of the lateral surface formation: a) Lateral surface with internal vertex G b) Lateral surface with external vertex G c) Toroidal surface that envelopes these polyhedral structures



a)



b)

Figure 4. a) The unit hexahedral element ABCDEFG with the internal vertex G b) Figure 4.b The unit hexahedral element ABCDEFG with the external vertex G

2.1 Descriptive geometric interpretation

From the descriptive geometry's point of view, we will analyze a spatial interpretation of the vertices position of one cell in the lateral surface of CA II, the spatial hexahedral element ABCDEFG. This element is rested by its edge AB on the side of the base polygon, which we adopted to lie in the horizontal plane (x-y). As the upper basis of this concave polyhedron is parallel to the lower, the top edge DE of the spatial hexahedral element, will also be horizontal and will be located at twice the height of the central vertex G, because the element itself is symmetrical in relation to the plane α and also in the relation to the horizontal plane γ at the level of the vertex G. Vertices C and F will be in the same plane γ as the vertex G. Vertices B, C and D are located in the plane β. If we determine the position of any of the vertices G, C or F, we have solved the problem of the position of all the other vertices.

Suppose that the point C of the spatial hexahedral element ABCDEFG is located on the sphere L with radius r=a (side of the equilateral triangle) centered at the vertex B. The point G and A belong to the same sphere. The vertex G is set on the circle of the radius $R1=\frac{a}{2}\sqrt{3}$ in the plane α, which represent the circular trajectory of rotation for the vertex G (as a vertex of the equilateral triangle would rotate around one of its sides, AB, by the circle of the radius equal to the triangle's altitude). If we assume that G is the center of a horizontal circle c of radius R_2=a, on which the vertices C and F lie, the movement of the vertex G by the trajectory R1, from the position G_0 to G_n, all associated circles c_0-c_n will form a circular quartic surface B - Bohemian dome (Fig. 5). By the intersection of the quartic surface B and the sphere L, we obtain the curve of eighth order (Fig. 6), which will represent the trajectory of the vertex C during the mechanical movement of spatial hexahedron ABCDEFG. Having the double tangential plane in the common tacnode point A, as well as the two more simple tangential planes, this curve will degenerate into the sixth order curve t and the circle. For our consideration, in the interpretation of the vertex C trajectory and finding the position of the point C, only the sixth order curve will be of the interest, while the circle will not participate in the solution. The curve t is the geometrical locus of the vertices c_0-c_n positions for the initial position of the vertex G.

The paper [12] gives a constructive-geometric procedure for finding the position of the vertex C, based on the above.

The Fig. 7 shows the spatial trajectory t obtained as the extracted intersection curve of the sphere L and the Bohemian Dome B. The trajectory t of the vertex C is given in axonometric view. It can be constructively generated by an iterative graphical method, starting from the initial height h=0 (C_0=A) to maximum height ($G_{max}C_{max}$). We can see that the plane β intersect the trajectory t in the four points: two above and two below the plane x-y in which the lower base of CA II is settled. This corresponds to the above insight, that for the same basis we obtain two variations of CA II: one with the internal and one with the external vertex G. Hence, the observed base polygon can be adopted for the upper, as well as for the lower basis. When the vertex G is internal point of the hexahedral element, the vertex C is external, and vice versa.

Since we have determined the height of the vertex C, we have found the heights of all the vertices, knowing that hc=hf=hg, and hd=he=2hc, i.e. that any hexahedral element of CA II is plane symmetrical, in relation to the plane γ of the vertex G.

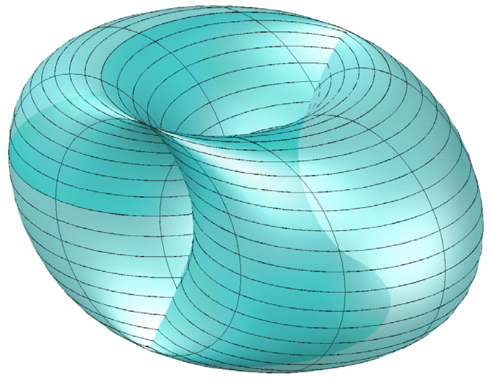


Figure 5. Bohemian dome

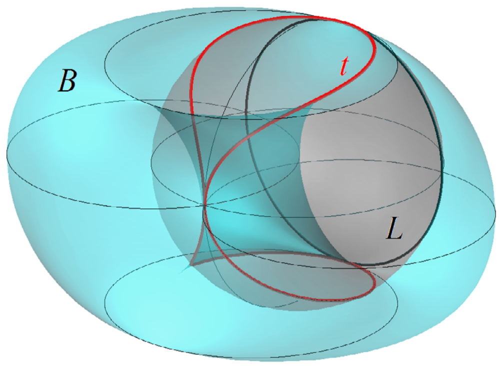


Figure 6. Bohemian Dome (B) and sphere (L) intersection – the trajectory (t) of the vertex C

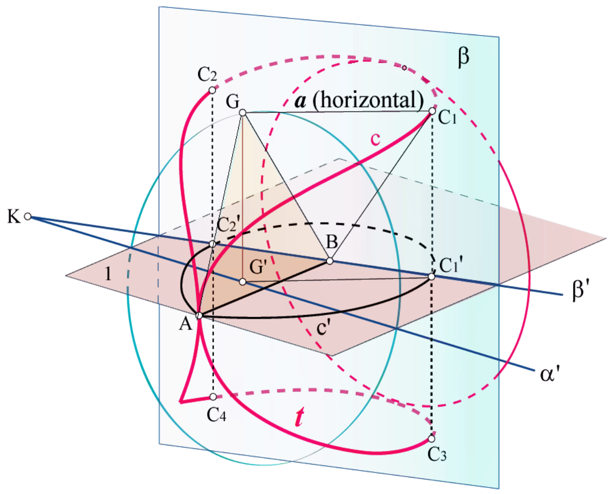


Figure 7. The spatial model of the vertex C trajectory

2.2 Iterative Numerical Method and Setting the Algorithm

The problem of getting the height of CA II is hence reduced to the intersection of the plane β and space curve *t* of the sixth order. A problem of sixth order is not exactly geometrically solvable by the classical accessories, compasses and straightedge, and a set of analytical equations would require a complex algebraic notation, quite challenging from the mathematical point of view, but unnecessary for engineering purposes. On the other hand, by the application of iterative numerical methods and by setting the appropriate algorithms, the values of the required parameters are obtained with a satisfactorily negligible error (for the iteration, the authors applied Microsoft Excel). The existing error appears eventually in the seventh decimal place for the assumed edge size a= 100.

The Algorithm (given in the Table 1) set according to Fig. 8 for finding the parameters of CA II-M differs from the algorithm for the CA II-m only in formulas (10) and (13). For the input data a and n, and the output H, the algorithm gives the tabulated results of the parameters, when the condition $\Delta_{(H)}$=0 is fulfilled.

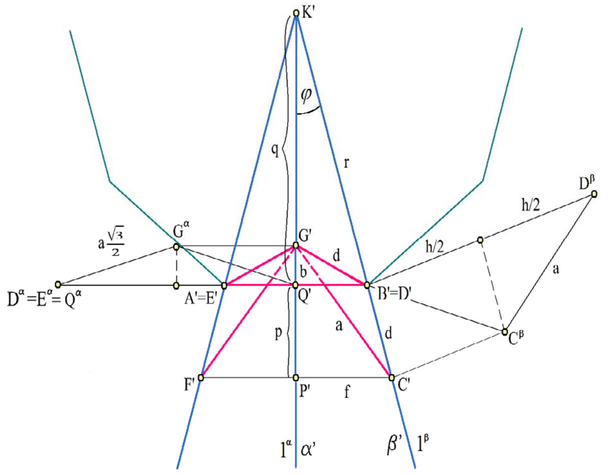


Figure 8. The parameters and metric relations in the spatial hexahedral element ABCDEFG

In the Table 1, the examples of the calculated measurements are given for CA II-M-16, and CA II-m-16, CA II-m-32 and CA II-m-132, in order to compare them. The above Algorithm will show that for the variant CA II-M with the internal vertex G, the heights slightly decreases with increasing the number of basic polygon's sides, from n=3 to n=∞, and for the variant CA II-m the height increases, from n=5 to the final height of basis n = ∞ which will be equal to: $hn_{\infty}=1,632663a$

This will be the same height hn_{∞} for both the variants CA II-M and CA II-m, since in this case, both the variants reduces to the same model, only viewed from the opposite sides, as shown in Fig. 9 and Fig. 11**.**

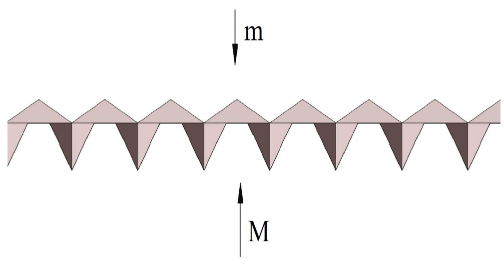


Figure 9. Concave antiprism of second sort (CA-II) with n=∞

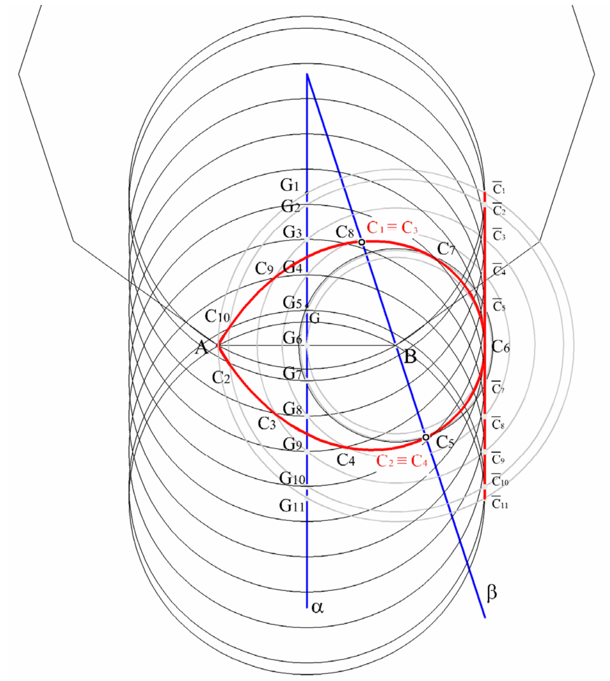


Figure 10. Top view on the trajectory curve

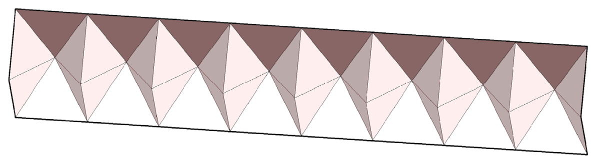


Figure 11. Concave antiprism of second sort (CA-II) with n=∞, axonometric view

2.3 The Method of mechanical model

The spatial hexahedral element ABCDEFG acts as a mechanism [13], which characteristics we present in this section of the paper. We need to define the movement of this mechanism, with respect to the above conditions.

The geometrical principles exposed in this paper can be interpreted mechanically and realized by one particularly synthesized mechanism. Top and oblique projection of that mechanism are shown in Fig.12 [12].

As explained before, the trajectory of the vertex C is the spatial sextic curve. In the top view, because of its plane symmetry, it is projected as the cubic curve [12], and the vertical plane β is projected as a line. The problem is reduced to the intersection of the line and the projected curve t', which will be used in the seting the appropriate mechanism.

The mechanism comprises the revolute joints R and M, prismatic joints P and S, spherical joint G, arm p, lever GC and crank k. The mobility (degree(s) of freedom) of this linkage system is 1. Prismatic joint P can slide along the groove of the arm p, and arm p can rotate about the axis of the revolute joint R. Prismatic joint P holds the spherical joint G, in which the lever GC is attached. Crank k is always held in horizontal position by the revolute joint M and lever GC is connected with the crank k under the right angle by the prismatic joint S in such a way that geometrical center of joint S is mid point of the lever GC. Thus, lever GC is also held in horizontal position by its joints and crank

k. The complete mechanism is driven by a rotary motor in the point B which rotates the crank k about the vertical axis of the revolute joint M. During the mechanism motion, center of the spherical joint G remains in vertical plane α and the point C generates the plane curve. The mechanism motion lasts until the lever GC touches the surface of the vertical panel β. This contact point is the intersection point between trajectory t of the point C and plane β and represents the required solution of the geometrical problem modeled by this mechanism.

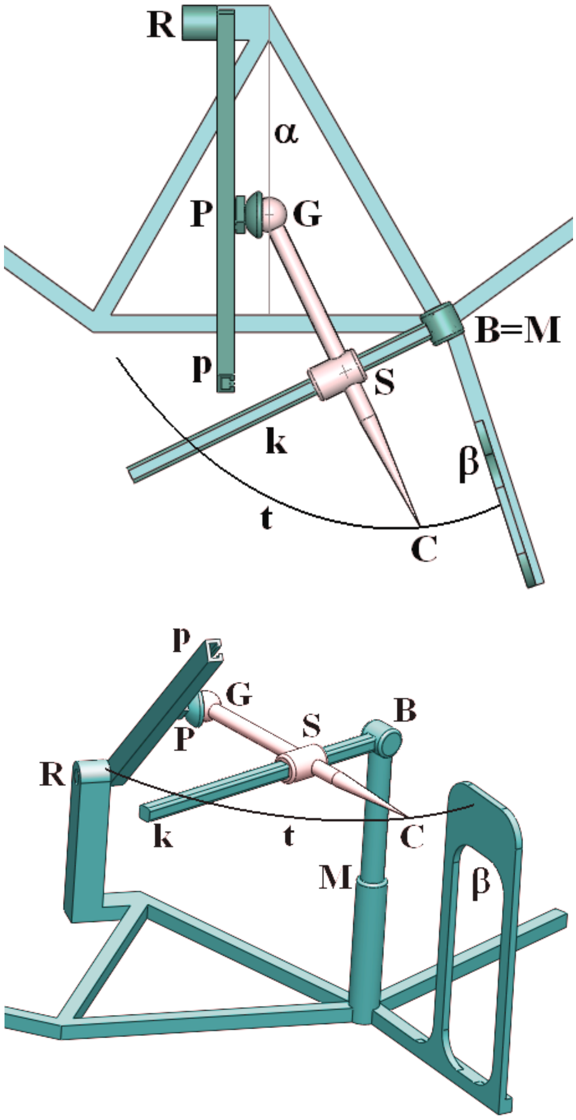


Figure 12. The mechanism that describes planar cubic curve, the plane projection of the trajectory curve from the Fig. 10.

3D model of this mechanism is accomplished by the using of Solid Works application.

3. BEHAVIOR OF CA II ON THE STATIC AND DYNAMIC EFFECTS

In order to investigate the potential application of these polyhedral structures in engineering, testing was performed on the static and dynamic effects for one representative of the observed family of polyhedra: concave antiprism of second sort, with the base polygon having n=16 sides: CA II-M-16. We investigated the rigidity of the structure and compared its behavior with the regulations for our seismic areas.

CA II-M-16 is realized in this analysis by the system of pipes, 10^{10} N/mm^2, density: 2800 kg/m^3, yield strength: 75,829 MPa). The adopted pipe length is 7,2m, made of 2024 aluminum alloy (elasticity modulus: 7,3 the diameter is 200 mm and the pipe wall thickness is 10 mm). All the pipes are connected by the spherical joints.

It is important to indicate that the entire structure must have fixed bases - hexadecagons, as it guarantees the geometric determination of the structure.

The framework has fixtures at the sixteen bottom spherical joints, so the complete structure is exclusively subjected to the force of gravity, i.e. to the framework self weight. Structural static analysis has been accomplished by using SolidWoks 2012 application and the following results have been obtained:

Stress (Fig. 13-a); max. stress: 5,6 MPa

Displacement (Fig. 13-b); max. displace-ment: 2,357 mm.

Strain (Fig. 13-c); max. strain: 5,587 10^{-5}.

By using the same application subroutines, first three mode shapes of oscillations and corresponding resonant frequencies have been determined for the same framework structure. The results of this dynamic analysis are particularly significant since they can disclose and clarify the structure vulnerability to the impact of seismic waves. The first five mode shapes of oscillations are shown in Fig. 13-d to Fig. 13-h, and the corresponding resonant frequencies are: 9,2247Hz, 9,339Hz, 9,3794Hz, 9,4124Hz, 9,4125Hz, respectively.

Regarding the static analysis, the following conclusions can be drawn:

The maximal structural stress generated by the framework self weight is over 13 times less than the material yield strength - the framework structure has significant strength.

From the fact that the maximal displacement and strain of the structure are extremely small it can be concluded that the structure possesses great stiffness.

Dynamic analysis also offers some important conclusions:

Since the frequencies of the first and the all the other oscilation modes are significantly higher than the frequency of the maximal acceleration frequencies (0,3Hz - 5Hz) of the typical regional and local seismic motions, the risk of structure seismic resonance is completely avoided.

The relatively high mode shapes eigenvalues of oscillations of this huge framework confirm the already emphasized fact that this structure possesses high stiffness, achieved by the specific geometrical form of the hexadecagonal concave antiprism of the second sort.

All these considerations and conclusions prove that constructions based on the hexadecagonal concave antiprism of the second sort are highly convenient for,engineering purposes, thanks to its geometrical properties.

Note: All the diagrams presented above, show multiply increased deformation, so that they could be visible.

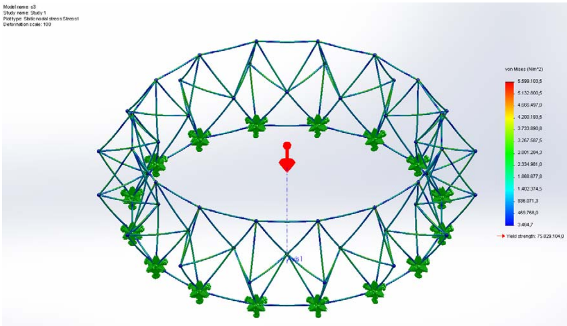


Figure 13.a Static Node Stress 1

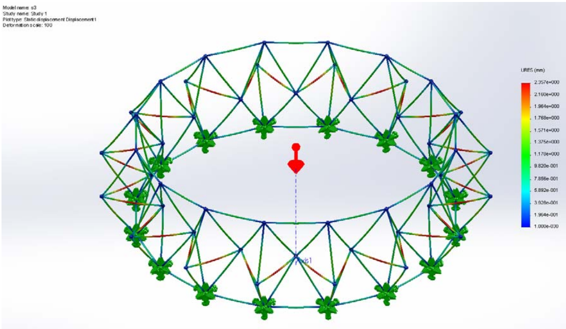


Figure 13.b Static Displacement

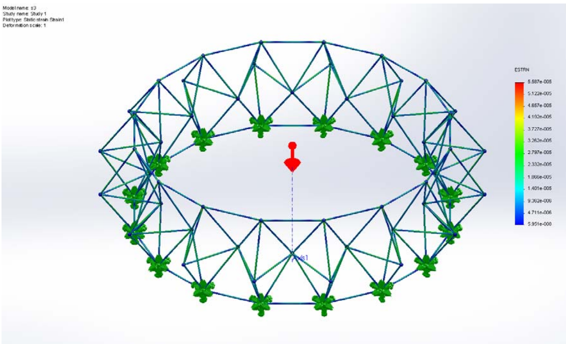


Figure 13.c Static Strain

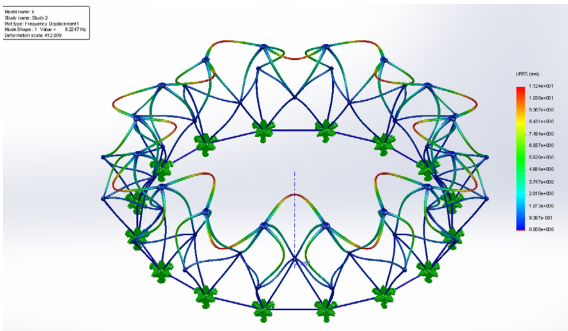


Figure 13.d Frequency Displacement 1

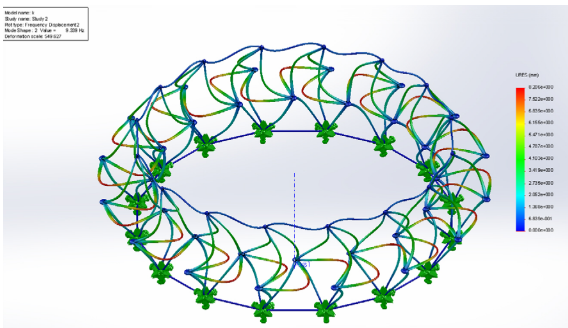


Figure 13.e Frequency Displacement 2

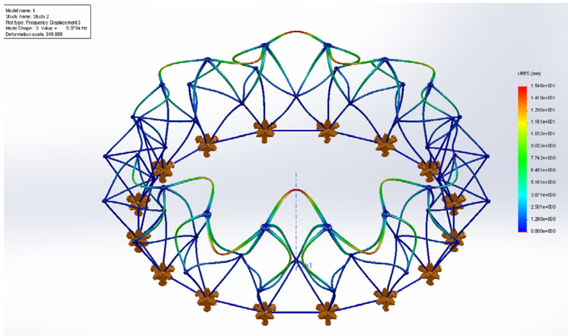


Figure 13.f Frequency Displacement 3

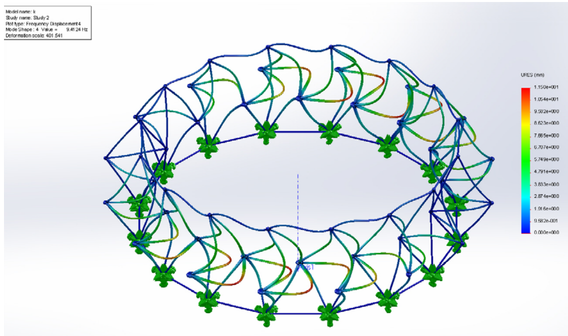


Figure 13.g Frequency Displacement 4

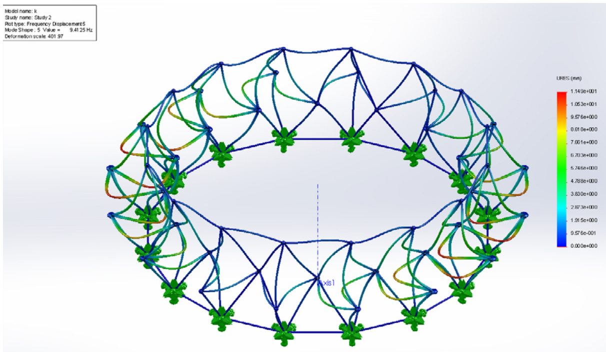


Figure 13.h Frequency Displacement 5

Behaviour of CA II-16-M on the static and dynamic effects is given in Figures 13a to 13h.

4. CONCLUSIONS

After completing the examinations of CA II from geometric, mathematical, mechanical and structural point of view, we draw the following conclusions:

The polyhedral structures, concave antiprism of second sort (CA II) are geometrically wholly defined, and for each given n it is possible to determine accurately and uniquely all the necessary measures, linear and angular parameters, as well as the position and attitudes of all the vertices.

Heights of these polyhedra are very slightly changed depending on the number of the base sides, ending with the final height of the CA II "strip" with the infinite number of base sides, as proven by the given algorithm.

CA II are regular faced concave polyhedra, with all the edges equivalent, so they can be modeled by identical sticks.

By its geometric distinctness, these structures are rigid and statically stable.

Because of its rigidity, these structures are resistant to external influences, pressure, tension, and the seismic effects, and can be used for various engineering purposes: the piston design, spatial grids, and (let us use some ideas from [3]) undersea observatory, submarine underwater base, as an orbiting space station, as a land-based liquid storage vessel, and as an undersea nuclear reactor housing.

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О ОСОБИНАМА КОНКАВНИХ АНТИПРИЗМИ ДРУГЕ ВРСТЕ

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Рад се бави испитивањем геометријских, статичких и динамичких особина једне полиедарске структуре настале набирањем дворедног сегмента мреже једнакостраничних троуглова. Основе ових конкавних полиедара су правилни, идентични полигони у паралелним равнима, повезани низом наизменичних троуглова, као и у случају конвексних антипризми. Постоје две варијанте савијања овакве мреже, па самим тим и два типа конкавних антипризми друге врсте (KA II) за сваку посматрану основу од n=5, n=∞. У раду су размотрени начини добијања тачног положаја темена и других линеарних параметара ових полиедара, уз примену алгоритма за њихово математичко израчунавање. Структурална анализа једног представника ових полиедара дата је коришћењем апликација програма SolidWorks, како би се испитала могућност примене ових облика у инжењерству.

Table 1. Algorithm for finding the parameters of CA II, and numerical values of the parameters for some n

	Algorithm: Input : a, n Output: H	Values for n, (variant M)	Values for n, (variant m)		
1	(n) number of the base polygon sides	n=16	n=16	n=32	n=132
2	(h) CA II height	166,0363692	159,7979133	161,6575103	162,9197086
3	(a) side of the base polygon	a=100	a=100	a=100	a=100
4	$\varphi = \frac{\pi}{n}$	$11^o 15'$	$11^o 15'$	5^0 37'30''	1^0 21'49''
5	$K'A' = K'B' = r = \frac{a}{2\sin\varphi}$	256,2915448	256,2915448	510,1148619	21,1043595
6	$K'Q' = q = r \cdot \cos\varphi$	251,3669746	251,3669746	507,6585194	210,0448568
7	$Q^\alpha G^\alpha = \frac{a}{2}\sqrt{3}$	86,6025404	86,6025404	86,6025404	86,6025404
8	$G'Q' = b = \frac{1}{2}\sqrt{3a^2 - h^2}$	24,6572713	33,4089318	31,091998	29,3988458
9	$C'B' = B'G' = d = \frac{1}{2}\sqrt{4a^2 - h^2}$	55,7492693	60,1344887	58,8787936	58,0025184
10	$K'C' = e = r + d$ *for the variation a)*	312,0408141			
11	$K'C' = e = r - d$ *for the variation b)*		196,1570561	451,2360682	2043,041076
12	$P'C' = f = e \cdot \sin\varphi$	60,8761429	38,2683432	44,228869	48,6196736
13	$P'Q' = p = \sqrt{e^2 - f^2} - q$ *for the variation a)*	54,6780627			
14	$P'Q' = p = q - \sqrt{e^2 - f^2}$ *for the variation b)*		58,9790213	58,5952762	57,9860917
15	$G'C' = a_1 = \sqrt{(p+b)^2 + f^2}$	100,0000000	100,0000000	100,0000000	99,9999999
16	$\Delta = a - a_1$	Δ=0.0000000	Δ=0.0000000	Δ=0.0000000	Δ=0.0000000