METODA DINAMIČKE KRUTOSTI U ANALIZI VIBRACIJA KRUŽNE CILINDRIČNE LJUSKE

DYNAMIC STIFFNESS METHOD IN THE VIBRATION ANALYSIS OF CIRCULAR CYLINDRICAL SHELL

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1 UVOD

Ljuske se često koriste kao elementi inženjerskih konstrukcija, naročito u mašinstvu, građevinarstvu, avio i brodskom inženjeringu. Razlog je u tome što ljuske mogu da se koriste za konstrukcije velikih raspona, bez potrebe za međuosloncima, i što ovakve konstrukcije imaju povoljan odnos krutosti i težine. Ljuske se koriste za konstrukcije rezervoara, delova trupova aviona, delova konstrukcija brodova i slično. Tokom životnog veka, odnosno tokom njihove primene, ove konstrukcije izložene su složenim uslovima okruženja koji podrazumevaju najrazličitije granične uslove i opterećenja, kao što su npr. dinamički uticaji velikog intenziteta koji mogu da dovedu do kolapsa konstrukcije. Potpuno poznavanje dinamičkih karakteristika ovih konstruktivnih elemenata od velike je važnosti da bi se obezbedilo sigurno, uspešno i ekonomski isplativo projektovanje.

Metod dinamičke krutosti (MDK), koji u poslednje vreme privlači pažnju sve većeg broja istraživača, predstavlja alternativu metodu konačnih elemenata (MKE) u analizi vibracija. Kao što je poznato, veličina konačnog elementa mora da bude manja od petine talasne dužine koja odgovara najvišoj frekvenciji u analizi da bi se dobili rezultati zadovoljavajuće tačnosti. Zato proračun konstrukcija primenom MKE u oblasti visokih frekvencija, koje se javljaju u akustici ili kod vibracija mašina visokih frekvencija, postaje glomazan i

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1 INTRODUCTION

Shells are often used as elements of engineering structures, particularly in mechanical, civil, aerospace and naval engineering. The reason is that the shells can be used for a large-span structures, without intermediate supports, and because they have a favourable stiffness to weight ratio. Shells are used for the construction of reservoirs, pressure vessels, fuselage, naval vehicles, etc. During their lifetime and exploitation, these structures are subjected to various loadings under complex boundary conditions, such as violent dynamic load, that can lead to collapse of structures. Therefore, a complete knowledge of the dynamic characteristics of these structural elements is very important to ensure a safe, successful and economically feasible design.

The dynamic stiffness method (DSM), which has recently been attracting the attention of a growing number of researchers, is an alternative to the finite element method (FEM) in the vibration and buckling analysis. As known, the size of the finite elements must be less than a fifth of the wavelength corresponding to the highest frequency in the analysis, in order to obtain satisfactory results. Therefore, the analysis of structures by FEM in the high frequencies range, that occur in acoustics and vibrations of machine at high frequencies, becomes cumbersome and time consuming because of the need for condensing the finite element mesh.

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dugotrajan zbog potrebe za progušćenjem mreže konačnih elementa.

Dinamička matrica krutosti, koja povezuje sile i pomeranja na granicama elementa, dobija se na osnovu tačnog rešenja parcijalnih diferencijalnih jednačina kojima je definisan problem slobodnih vibracija u frekventnom domenu. Samim tim, matrice krutosti određene na ovaj način jesu tačne, frekventno zavisne i sadrže, pored krutosti, inerciju i prigušenje, pa se elementi zasnovani na tim matricama nazivaju i kontinualni elementi. Njihovom primenom diskretizacija domena je svedena na minimum, odnosno jedan kontinualni element je dovoljan da se opiše ponašanje konstruktivnog elementa konstantnih materijalnih i geometrijskih karakteristika. Globalna dinamička matrica krutosti formira se na sličan način kao i u MKE, s tom razlikom što su u MDK dvodimenzionalni elementi, umesto u čvorovima, spojeni duž kontura. Na taj način, primenom MDK moguća je analiza i konstruktivnih sistema. Na osnovu svega rečenog, može da se zaključi da je MDK spojio prednosti analitičkih i diskretnih-numeričkih (MKE) metoda.

Dinamičke matrice krutosti prvo su bile izvedene za jednodimenzionalne elemente. Na osnovu rešenja problema slobodnih vibracija pravog grednog nosača, koje postoji u zatvorenom obliku, izvedene su dinamičke matrice krutosti po Euler–Bernoulli-jevoj i Timoshenko-ovoj teoriji [1-4]. Casimi*r* [5] je odredio dinamičku matricu krutosti zakrivljene grede, pri čemu je rešenje diferencijalne jednačine odredio numerički. Na isti način, Le Sourne [6] odredio je dinamičku matricu krutosti kružne cilindrične ljuske. Kontinualni elementi formulisani na ovakav način nazivaju se i numerički kontinualni elementi.

Kod pravougaone ploče, problem slobodnih vibracija može da se reši u zatvorenom obliku samo za specijalne granične uslove. Gorman [7] je predložio analitičko rešenje problema slobodnih vibracija pravougaone ploče, koje je nezavisno od graničnih uslova, u obliku sume jednostrukih Fourier-ovih redova. Ovaj metod se naziva metod superpozicije i korišćen je, uz metod projekcije kojim su diskretizovani granični uslovi, za izvođenje dinamičke matrice krutosti pravougaone ploče po Kirchhoff-oj teoriji [8]. Primenom metoda superpozicije i metoda projekcije formulisana je i dinamička matrica krutosti pravougaone ploče za naprezanje u ravni [9], za poprečne vibracije po Mindlin-ovoj teoriji [10], kao i za poprečne vibracije slojevite ploče primenom smičuće teorije višeg reda [11].

Ako je debljina ljuske manja od 1/20 talasne dužine i/ili poluprečnika krivine, teorija tankih ljuski, gde su zanemarene smičuća deformacija i rotaciona inercija, generalno je prihvatljiva. U zavisnosti od pretpostavki učinjenih tokom izvođenja kinematičkih relacija, definisanja presečnih sila i uslova ravnoteže, formulisane su različite teorije tankih ljuski, čiji je pregled dat u knjizi od Leisse-a [12]. U ovom radu je formulisana dinamička matrica krutosti kružne cilindrične ljuske po Flügge-ovoj teoriji. U sistemu parcijalnih diferencijalnih jednačina, kojima je definisan problem slobodnih vibracija, izvodi po vremenu su eliminisani korišćenjem spektralne dekompozicije, dok je tangencionalna koordinata eliminisana primenom rešenja u obliku Fourier-ovog reda. Na taj način je dobijen sistem od tri obične diferencijalne jednačine koje su funkcija samo

The formulation of the dynamic stiffness (DS) matrix, which relates forces and displacements at the boundaries of the element, is based on the exact solutions of the free vibration problem in the frequency domain. Therefore, the stiffness matrices determined in this manner are accurate, frequency dependent, and contain among the stiffness, inertia and damping. Therefore, the elements based on these matrices are called continuous elements. By their usage, the discretization of observed domain is reduced to a minimum, i.e. one continuous element is sufficient to describe behaviour of a structural element of constant material and geometrical properties. The global stiffness matrix is obtained using the assembly procedure in the same way as the FEM, with one exception - in the DSM two dimensional elements, instead at nodes, are connected along the boundary lines. Finally, it can be concluded that the DSM uses advantages of analytical as well as numerical methods (FEM).

The dynamic stiffness matrix was first developed for one-dimensional elements. Based on the closed form solution of the free vibration of a straight beam, the dynamic stiffness matrix according to the Euler-Bernoulli and Timoshenko beam theory was formulated [1-4]. Casimir [5] developed the dynamic stiffness matrix of a curved beam, whereby the differential equations were solved numerically. In the same way, Le Sourne [6] derived the dynamic stiffness matrix of a circular cylindrical shell. Continuous elements formulated in this way are called numerical continuous elements.

For a rectangular plate, the free vibration problem can be solved in a closed form only for special combinations of boundary conditions. Gorman [7] applied the method of superposition and obtained analytical solution for free vibration of a rectangular Kirchhoff plate with arbitrary boundary conditions, in the form of the Fourier series. Casimir [8] used the method of superposition, along with the method of projection for discretization of boundary conditions, and derived dynamic stiffness matrix of a rectangular Kirchhoff plate [8]. In the same manner, continuous rectangular plate elements for the in-plane vibration [9], transverse vibration according to the Mindlin plate theory [10] and transverse vibration of a laminated plate using the higher-order shear deformation theory [11] have been formulated, as well.

Thin shell theory, which neglects shear deformation and rotary inertia, is generally acceptable if the shell thickness is less than $1/20^{th}$ of the wavelength and/or radius of curvature. Depending on the assumptions made in the strain–displacement relations, the force and moment resultant equations and equilibrium equations, various thin shell theories have been formulated. A comprehensive overview of the shell theories is given in the book of Leissa [12]. In this paper the dynamic stiffness matrix of a circular cylindrical shell element according to the Flügge thin shell theory is formulated. In the governing partial differential equations time derivatives are eliminated by using the spectral decomposition, while the circumferential coordinate is eliminated by applying the solution in the form of Fourier series. Therefore, the governing system of three partial differential equations is transformed into the system of three ordinary differential equations, which depends only on the longitudinal coordinate. By expanding the podužne koordinate. Razvijanjem determinante ovog sistema jednačina dobijena je jedna obična diferencijalna jednačina osmog reda s konstantnim koeficijentima, čije je rešenje dobro poznato. Povezivanjem komponenti vektora sila i vektora pomeranja na krajevima ljuske formirana je dinamička matrica krutosti, koja je implementirana u za tu svrhu napisani Matlab [13] program za određivanje sopstvenih vrednosti i oblika oscilovanja kružne cilindrične ljuske. Metod dinamičke krutosti primenjen je u analizi slobodnih vibracija kružne cilindrične ljuske s različitim kombinacijama graničnih uslova, skokovitom promenom debljine, kao i s prstenastim međuosloncima. Putem numeričkih primera izvršena je verifikacija dobijenih rezultata poređenjem s dostupnim analitičkim rešenjima u literaturi, kao i s rezultatima Abaqus-a [14].

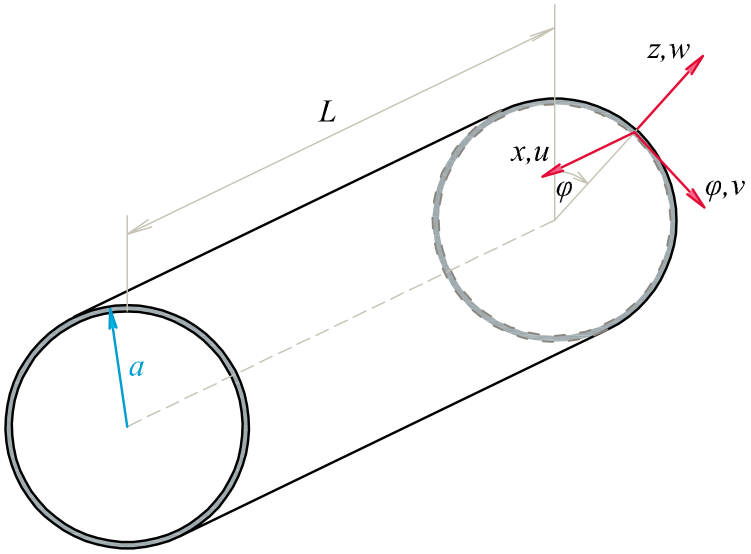
determinant of this system of equations, one eighth order ordinary differential equation with constant coefficients is obtained. The solution of this equation is well known. The dynamic stiffness matrix is formulated by relating the force and displacement vectors at the ends of a circular cylindrical shell element. It is implemented in the computer program, written in Matlab [13], for computing natural frequencies and mode shapes of circular cylindrical shell assemblies. The dynamic stiffness method is applied to the free vibrations analysis of circular cylindrical shells with different combinations of boundary conditions, stepped thickness variation and intermediate ring supports. Verification of the obtained results is carried out in comparison with available analytical solutions in the literature, as well as with the results of the Abaqus [13].

2 FORMULACIJA PROBLEMA

Na slici (1) prikazana je zatvorena kružna cilindrična ljuska konstantne debljine *h*, poluprečnika *a* i dužine *L*. Sa u, v i w označena su komponentalna pomeranja srednje površi ljuske.

2 FORMULATION OF THE PROBLEM

Figure (1) shows a closed circular cylindrical shell of constant thickness *h*, radius *a* and length *L*, where *u*, *v* and *w* are the displacements of the mid surface in *x*, φ and *z* directions, respectively.



Slika 1. Geometrija i koordinatni sistem zatvorene kružne cilindrične ljuske
Fig. 1. Geometry and coordinate system for a closed circular cylindrical shell

2.1 Osnovne jednačine slobodnih vibracija kružne cilindrične ljuske prema Flügge-ovoj teoriji tankih ljuski

Problem slobodnih vibracija kružne cilindrične ljuske po Flügge-ovoj teoriji definisan je sledećim sistemom parcijalnih diferencijalnih jednačina [12]:

2.1 Governing differential equations for free vibration of a circular cylindrical shell according to the Flügge thin shell theory

The governing differential equations of the Flügge thin shell theory are [12]:

$$\begin{bmatrix} \partial_x^2 + a_1\partial_\varphi^2 + a_2\partial_t^2 & a_3\partial_x\partial_\varphi & a_4\partial_x + a_5\partial_x^3 + a_6\partial_x\partial_\varphi^2 \\ a_3\partial_x\partial_\varphi & a_7\partial_\varphi^2 + a_8\partial_x^2 + a_2\partial_t^2 & a_7\partial_\varphi + a_9\partial_x^2\partial_\varphi \\ a_4\partial_x + a_5\partial_x^3 + a_6\partial_x\partial_\varphi^2 & a_7\partial_\varphi + a_9\partial_x^2\partial_\varphi & \begin{matrix} k\left(\partial_x^4 + 2a_7\partial_x^2\partial_\varphi^2 + a_7{}^2\partial_\varphi^4\right) \\ +a_7 - a_2\partial_t^2 + 2a_{10}\partial_\varphi^2 + a_{10} \end{matrix} \end{bmatrix} \begin{bmatrix} u(x,\varphi,t) \\ v(x,\varphi,t) \\ w(x,\varphi,t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (1)$$

gde su sa $u(x,\varphi,t)$, $v(x,\varphi,t)$ i $w(x,\varphi,t)$ označena pomeranja u aksijalnom, tangencijalnom i radijalnom pravcu, $\partial_x = d/dx$, $\partial_\varphi = d/d\varphi$, $\partial_t = d/dt$, $k = h^2/12$ i:

where $u(x,\varphi,t)$, $v(x,\varphi,t)$ and $w(x,\varphi,t)$ denote displacement components in the axial, tangential and radial direction, $\partial_x = d/dx$, $\partial_\varphi = d/d\varphi$, $\partial_t = d/dt$, $k = h^2/12$ and:

$$a_1 = \frac{1-\nu}{2a^2}\left(1+\frac{K}{Da^2}\right) \quad a_2 = -\frac{\rho h}{D} \quad a_3 = \frac{1+\nu}{2a} \quad a_4 = \frac{\nu}{a} \quad a_5 = -\frac{K}{Da}$$
$$a_6 = \frac{1-\nu}{2a^3}\frac{K}{D} \quad a_7 = \frac{1}{a^2} \quad a_8 = \frac{1-\nu}{2}\left(1+\frac{3K}{Da^2}\right) \quad a_9 = \frac{3-\nu}{2}\frac{K}{Da^2} \quad a_{10} = \frac{K}{Da^4} \quad (2)$$

U jednačinama (2) korišćene su sledeće oznake: ν je Poisson-ov koeficijent, $K = Eh^3/12\left(1-\nu^2\right)$ je krutost na savijanje ljuske, $D = Eh/\left(1-\nu^2\right)$ je krutost u „ravni" ljuske, ρ je gustina mase i E je Young-ov modul elastičnosti.

Izrazi za presečne sile, u funkciji komponentalnih pomeranja, jesu [12]:

In Eqs (2) ν is the Poisson's ratio, $K = Eh^3/12\left(1-\nu^2\right)$ is the flexural stiffness, $D = Eh/\left(1-\nu^2\right)$ is the stiffness in the mid surface of shell, ρ is the mass density and E is the Young's modulus of elasticity.

The expressions of forces and moments in terms of the displacements are [12]:

$$N_x = D\left[\frac{\partial u}{\partial x} + \frac{\nu}{a}\left(w + \frac{\partial v}{\partial \varphi}\right)\right] - \frac{K}{a}\frac{\partial^2 w}{\partial x^2} \qquad N_\varphi = D\left[\frac{1}{a}\left(w + \frac{\partial v}{\partial \varphi}\right) + \nu\frac{\partial u}{\partial x}\right] + \frac{K}{a^3}\left(\frac{\partial^2 w}{\partial \varphi^2} + w\right)$$
$$N_{x\varphi} = \frac{D(1-\nu)}{2}\left(\frac{1}{a}\frac{\partial u}{\partial \varphi} + \frac{\partial v}{\partial x}\right) + \frac{K(1-\nu)}{2a^2}\left(\frac{\partial v}{\partial x} - \frac{\partial^2 w}{\partial x\partial\varphi}\right)$$
$$N_{\varphi x} = \frac{D(1-\nu)}{2}\left(\frac{1}{a}\frac{\partial u}{\partial \varphi} + \frac{\partial v}{\partial x}\right) + \frac{K(1-\nu)}{2a^2}\left(\frac{1}{a}\frac{\partial u}{\partial \varphi} + \frac{\partial^2 w}{\partial x\partial\varphi}\right) \quad (3)$$
$$M_x = -K\left[\frac{\partial^2 w}{\partial x^2} + \frac{\nu}{a^2}\left(\frac{\partial^2 w}{\partial \varphi^2} - \frac{\partial v}{\partial \varphi}\right) - \frac{1}{a}\frac{\partial u}{\partial x}\right] \qquad M_\varphi = -K\left[\frac{1}{a^2}\left(\frac{\partial^2 w}{\partial \varphi^2} + w\right) + \nu\frac{\partial^2 w}{\partial x^2}\right]$$
$$M_{x\varphi} = -\frac{K(1-\nu)}{a}\left(\frac{\partial^2 w}{\partial x\partial\varphi} - \frac{\partial v}{\partial x}\right) \qquad M_{\varphi x} = -\frac{K(1-\nu)}{2a}\left(2\frac{\partial^2 w}{\partial x\partial\varphi} - \frac{\partial v}{\partial x} + \frac{1}{a}\frac{\partial u}{\partial \varphi}\right)$$

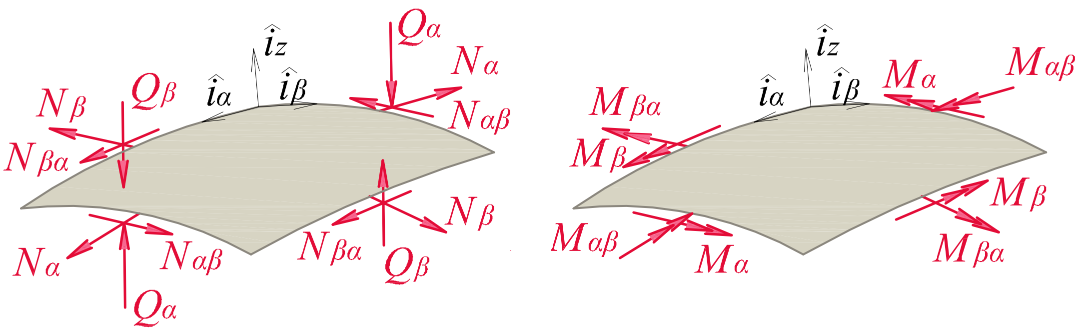
Transverzalne sile Q_x i Q_φ određene su iz uslova ravnoteže:

Shear forces Q_x and Q_φ are determined from the equilibrium conditions:

$$Q_x = \frac{\partial M_x}{\partial x} + \frac{1}{a}\frac{\partial M_{\varphi x}}{\partial \varphi} \qquad Q_\varphi = \frac{1}{a}\frac{\partial M_\varphi}{\partial \varphi} + \frac{\partial M_{x\varphi}}{\partial x} \quad (4)$$

Konvencija o pozitivnim presečnim silama data je na sledećoj slici:

The positive directions of the force and moment resultants are given in the next figure:



Slika 2. Konvencija o pozitivnim presečnim silama
Fig. 2. Positive directions of the force and moment resultants

2.2 Opšte rešenje problema slobodnih vibracija

Rešenje sistema jednačina (1) pretpostavljeno je u obliku proizvoda dve funkcije, pri čemu je jedna funkcija prostornih koordinata, a druga funkcija vremena:

2.2 General solution of the free vibration problem

By using the method of separation of variables, the general solution of the system of Eqs. (1) is sought in the following form:

$$u(x,\varphi,t)=\hat{u}(x,\varphi)e^{i\omega t} \qquad v(x,\varphi,t)=\hat{v}(x,\varphi)e^{i\omega t} \qquad w(x,\varphi,t)=\hat{w}(x,\varphi)e^{i\omega t} \tag{5}$$

U jednačini (5) ω predstavlja kružnu frekvenciju, dok su $\hat{u}$, $\hat{v}$ i $\hat{w}$ amplitude komponentalnih pomeranja u frekventnom domenu. Zamenom (5) u (1) dobija se:

In Eqs. (5) ω is the circular frequency, while $\hat{u}$, $\hat{v}$ and $\hat{w}$ are the amplitudes of componential displacements. Substituting Eqs. (5) into Eqs. (1) gives:

$$\begin{bmatrix} \partial_x^2 + a_1\partial_\varphi^2 - a_2\omega^2 & a_3\partial_x\partial_\varphi & a_4\partial_x + a_5\partial_x^3 + a_6\partial_x\partial_\varphi^2 \\ a_3\partial_x\partial_\varphi & a_7\partial_\varphi^2 + a_8\partial_x^2 - a_2\omega^2 & a_7\partial_\varphi + a_9\partial_x^2\partial_\varphi \\ a_4\partial_x + a_5\partial_x^3 + a_6\partial_x\partial_\varphi^2 & a_7\partial_\varphi + a_9\partial_x^2\partial_\varphi & \begin{array}{c} k\left(\partial_x^4 + 2a_7\partial_x^2\partial_\varphi^2 + a_7{}^2\partial_\varphi^4\right) \\ +a_7 + a_2\omega^2 + 2a_{10}\partial_\varphi^2 + a_{10} \end{array} \end{bmatrix} \begin{bmatrix} \hat{u}(x,\varphi) \\ \hat{v}(x,\varphi) \\ \hat{w}(x,\varphi) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{6}$$

Za zatvorenu kružnu cilindričnu ljusku $\hat{u}$, $\hat{v}$ i $\hat{w}$ moraju da zadovolje uslov periodičnosti u tangencijalnom pravcu, pa je rešenje sistema jednačina (6) usvojeno u obliku beskonačnog Fourier-ovog reda:

For a closed circular cylindrical shell, the displacement components $\hat{u}$, $\hat{v}$ and $\hat{w}$ should satisfy periodicity in the tangential direction; therefore the solution of the system of Eqs. (6) can be assumed in the form of infinite Fourier series:

1st part | *2nd part*

$$\begin{aligned} \hat{u}(x,\varphi) &= \sum_{m=0}^{\infty} U_m(x)\cos(m\varphi) + \sum_{m=1}^{\infty} U_m(x)\sin(m\varphi) \\ \hat{v}(x,\varphi) &= \sum_{m=1}^{\infty} V_m(x)\sin(m\varphi) + \sum_{m=0}^{\infty} V_m(x)\cos(m\varphi) \\ \hat{w}(x,\varphi) &= \sum_{m=0}^{\infty} W_m(x)\cos(m\varphi) + \sum_{m=1}^{\infty} W_m(x)\sin(m\varphi) \end{aligned} \tag{7}$$

gde m predstavlja ceo broj. U slučaju da granični uslovi ne zavise od koordinate φ rešenja za pojedine harmonike su međusobno nezavisna, pa umesto rešenja u obliku sume, može da se posmatra rešenje samo za m-ti harmonik. U radu će biti prikazano rešenje za asimetrične vibracije ($m \geq 1$), dok rešenje za slučaj rotaciono-simetričnih vibracija ($m = 0$) nije razmatrano. Takođe, biće prikazan postupak za rešenje problema

where m is an integer. If the boundary conditions do not depend on φ, the solutions for the different harmonics are uncoupled. Therefore, instead of the solution in the form of the sum, only the solution for the m^{th} harmonic can be considered. In this paper the solution for the asymmetric vibration ($m \geq 1$) will be presented, while the solution for the case of axisymmetric vibration ($m = 0$) will not be discussed. Also, only the solution slobodnih vibracija za slučaj kada je usvojen 1. deo rešenja, jednačina (7), dok se rešenje za 2. deo određuje na isti način. Treba istaći da su sopstvene frekvencije koje se dobijaju za 1. i 2. deo rešenja iste, što znači da su kod zatvorene kružne cilindrične ljuske sve sopstvene frekvencije dvostruke.

Izbor trigonometrijskih funkcija u 1. delu rešenja, jednačina (7), omogućava transformaciju sistema jednačina (6) u sistem tri obične diferencijalne jednačine:

procedure for the 1st part of Eq. (7) will be presented, since the solution for the 2nd part can be obtained in the same way. It should be noted that the natural frequencies obtained for the 1st and 2nd part of Eq. (7) are the same, which means that for the closed circular cylindrical shell all the natural frequencies are double.

The choice of trigonometric functions in the 1st part of the solution, Eq. (7), allows transformation of the system of Eqs. (6) into the following system of three ordinary differential equations:

$$\begin{bmatrix} c_{1,m}\partial_x^2 + c_{2,m} & c_{3,m}\partial_x & c_{4,m}\partial_x^3 + c_{5,m}\partial_x \\ -c_{3,m}\partial_x & c_{6,m}\partial_x^2 + c_{7,m} & c_{8,m}\partial_x^2 + c_{9,m} \\ c_{4,m}\partial_x^3 + c_{5,m}\partial_x & -c_{8,m}\partial_x^2 - c_{9,m} & c_{10,m}\partial_x^4 + c_{11,m}\partial_x^2 + c_{12,m} \end{bmatrix} \begin{bmatrix} U_m(x) \\ V_m(x) \\ W_m(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{8}$$

gde su:

where are:

$$\begin{array}{llll} c_{1,m} = 1 & c_{2,m} = -m^2 a_1 - \omega^2 a_2 & c_{3,m} = -m a_3 & c_{4,m} = -a_5 \\ c_{5,m} = a_4 - m^2 a_6 & c_{6,m} = a_8 & c_{7,m} = -m^2 a_7 - \omega^2 a_2 & c_{8,m} = -m a_9 \\ c_{9,m} = -m a_7 & c_{10,m} = k & c_{11,m} = -2km^2 a_7 & c_{12,m} = +a_7 + \omega^2 a_2 + \\ & & & k\left(m^4 a_7{}^2 - 2m^2 a_7 + a_7{}^4\right) \end{array} \tag{9}$$

Razvojem determinante, sistem jednačina (8) svodi se na jednu jednačinu osmog reda, koja važi za sve tri funkcije:

By expanding the determinant of the system (8), the following eighth order differential equation, valid for all three functions, is obtained:

$$\left(\partial_x^8 + a_{1,m}\partial_x^6 + a_{2,m}\partial_x^4 + a_{3,m}\partial_x^2 + a_{4,m}\right)\Psi = 0 \tag{10}$$

gde je $\Psi = U_m(x)$ ili $V_m(x)$ ili $W_m(x)$, a:

where $\Psi = U_m(x)$ or $V_m(x)$ or $W_m(x)$ and:

$$\begin{aligned} a_{1,m} &= \frac{c_{10,m}\left(c_{3,m}^2 + c_{2,m}c_{6,m} + c_{1,m}c_{7,m}\right) + c_{1,m}\left(c_{11,m}c_{6,m} + c_{8,m}^2\right) + 2c_{4,m}\left(c_{3,m}c_{8,m} - c_{5,m}c_{6,m}\right) - c_{4,m}^2 c_{7,m}}{c_{6,m}\left(c_{1,m}c_{10,m} - c_{4,m}^2\right)} \\ a_{2,m} &= \frac{c_{10,m}\left(c_{1,m}c_{5,m} + c_{2,m}c_{7,m}\right) + c_{11,m}\left(c_{3,m}^2 + c_{2,m}c_{6,m} + c_{1,m}c_{7,m}\right) + c_{2,m}c_{8,m}^2 - c_{5,m}^2 c_{6,m}}{c_{6,m}\left(c_{1,m}c_{10,m} - c_{4,m}^2\right)} \\ &+ 2\frac{c_{3,m}\left(c_{5,m}c_{8,m} + c_{4,m}c_{9,m}\right) + c_{1,m}c_{8,m}c_{9,m} - c_{4,m}c_{5,m}c_{7,m}}{c_{6,m}\left(c_{1,m}c_{10,m} - c_{4,m}^2\right)} \\ a_{3,m} &= \frac{c_{7,m}\left(c_{2,m}c_{11,m} - c_{5,m}^2\right) + c_{12,m}\left(c_{3,m}^2 + c_{2,m}c_{6,m} + c_{1,m}c_{7,m}\right) + c_{1,m}c_{9,m}^2 + 2c_{9,m}\left(c_{2,m}c_{8,m} + c_{3,m}c_{5,m}\right)}{c_{6,m}\left(c_{1,m}c_{10,m} - c_{4,m}^2\right)} \\ a_{4,m} &= \frac{c_{2,m}\left(c_{12,m}c_{7,m} + c_{9,m}^2\right)}{c_{6,m}\left(c_{1,m}c_{10,m} - c_{4,m}^2\right)} \end{aligned} \tag{11}$$

Rešenje jednačine (10) usvaja se u obliku $\Psi = e^{rx}$ i dobija se karakteristična jednačina:

The solution of the equation (10) is sought in the form $\Psi = e^{rx}$ and the corresponding characteristic equation is obtained:

$$r^8 + a_{1,m}r^6 + a_{2,m}r^4 + a_{3,m}r^2 + a_{4,m} = 0 \tag{12}$$

Pomoću smene $\mu = r^2$ jednačina (12) redukuje se na sledeću jednačinu četvrtog stepena:

The eight order Eq. (12) can be reduced to the fourth order polynomial by substituting $\mu = r^2$:

$$\mu^4 + a_{1,m}\mu^3 + a_{2,m}\mu^2 + a_{3,m}\mu + a_{4,m} = 0 \tag{13}$$

čiji su koreni označeni sa $\mu_{i,m}$, $i = 1, 2, 3, 4$. Koreni jednačine (12) su:

whose roots are: $\mu_{1,m}$, $\mu_{2,m}$, $\mu_{3,m}$ and $\mu_{4,m}$. The roots of Eq. (12) are defined as:

$r_{1,m}=\sqrt{\mu_{1,m}}$, $r_{2,m}=-\sqrt{\mu_{1,m}}$, $r_{3,m}=\sqrt{\mu_{2,m}}$, $r_{4,m}=-\sqrt{\mu_{2,m}}$, $r_{5,m}=\sqrt{\mu_{3,m}}$, $r_{6,m}=-\sqrt{\mu_{3,m}}$, $r_{7,m}=\sqrt{\mu_{4,m}}$ i $r_{8,m}=-\sqrt{\mu_{4,m}}$.

Rešenja za nepoznate funkcije pomeranja su:

$r_{1,m}=\sqrt{\mu_{1,m}}$, $r_{2,m}=-\sqrt{\mu_{1,m}}$, $r_{3,m}=\sqrt{\mu_{2,m}}$, $r_{4,m}=-\sqrt{\mu_{2,m}}$, $r_{5,m}=\sqrt{\mu_{3,m}}$, $r_{6,m}=-\sqrt{\mu_{3,m}}$, $r_{7,m}=\sqrt{\mu_{4,m}}$ and $r_{8,m}=-\sqrt{\mu_{4,m}}$.

The solutions for unknown functions can be written in the following form:

$$U_m(x)=\sum_{i=1}^{8}A_{i,m}e^{r_{i,m}x}\qquad V_m(x)=\sum_{i=1}^{8}B_{i,m}e^{r_{i,m}x}\qquad W_m(x)=\sum_{i=1}^{8}C_{i,m}e^{r_{i,m}x}\tag{14}$$

pri čemu je samo osam integracionih konstanti, od ukupno 24 ($A_{i,m}, B_{i,m}, C_{i,m}$), međusobno nezavisno.

Ako se integracione konstante $A_{i,m}$ i $B_{i,m}$ izraze u funkciji $C_{i,m}$:

where only eight integration constants, of total 24 ($A_{i,m}, B_{i,m}, C_{i,m}$), are independent. If the integration constants $A_{i,m}$ and $B_{i,m}$ are expressed in terms of $C_{i,m}$:

$$A_{i,m}=\delta_{i,m}C_{i,m}\qquad B_{i,m}=\gamma_{i,m}C_{i,m}\tag{15}$$

gde su $\delta_{i,m}$ i $\gamma_{i,m}$ koeficijenti koji predstavljaju odnos amplituda aksijalnog i radijalnog, odnosno tangencijalnog i radijalnog pomeranja:

where $\delta_{i,m}$ and $\gamma_{i,m}$ are coefficients that represent the ratio of amplitudes of axial-radial and tangential-radial displacements respectively:

$$\begin{aligned}\delta_{i,m}&=\frac{\left(c_{9,m}+c_{8,m}\left(r_{i,m}\right)^2\right)^2+\left(c_{7,m}+c_{6,m}\left(r_{i,m}\right)^2\right)\left(c_{12,m}+c_{11,m}\left(r_{i,m}\right)^2+c_{10,m}\left(r_{i,m}\right)^4\right)}{r_{i,m}\left(c_{5,m}+c_{4,m}\left(r_{i,m}\right)^2\right)\left(c_{7,m}+c_{6,m}\left(r_{i,m}\right)^2\right)-c_{3,m}r_{i,m}\left(c_{9,m}+c_{8,m}\left(r_{i,m}\right)^2\right)}\\ \gamma_{i,m}&=\frac{c_{12,m}c_{3,m}+c_{5,m}c_{9,m}+\left(c_{11,m}c_{3,m}+c_{5,m}c_{8,m}+c_{4,m}c_{9,m}\right)\left(r_{i,m}\right)^2+\left(c_{10,m}c_{3,m}+c_{4,m}c_{8,m}\right)\left(r_{i,m}\right)^4}{c_{5,m}c_{7,m}-c_{3,m}c_{9,m}+\left(c_{5,m}c_{6,m}+c_{4,m}c_{7,m}-c_{3,m}c_{8,m}\right)\left(r_{i,m}\right)^2+c_{4,m}c_{6,m}\left(r_{i,m}\right)^4}\end{aligned}\tag{16}$$

dobijaju se analitički izrazi za komponentalna pomeranja kružne cilindrične ljuske u obliku:

the analytical expressions for displacement components are obtained in the following form:

$$\begin{aligned}\hat{u}(x,\varphi)&=\sum_{m=1}^{M}\left(\sum_{i=1}^{8}\delta_{i,m}C_{i,m}e^{r_{i,m}x}\right)\cos(m\varphi)\\ \hat{v}(x,\varphi)&=\sum_{m=1}^{M}\left(\sum_{i=1}^{8}\gamma_{i,m}C_{i,m}e^{r_{i,m}x}\right)\sin(m\varphi)\\ \hat{w}(x,\varphi)&=\sum_{m=1}^{M}\left(\sum_{i=1}^{8}C_{i,m}e^{r_{i,m}x}\right)\cos(m\varphi)\end{aligned}\tag{17}$$

Zamenom izraza (17) u jednačine (3) i (4) dobijaju se izrazi za presečne sile.

By substituting Eq. (17) into Eqs. (3) and (4), the expressions for forces and moments can be obtained, as well.

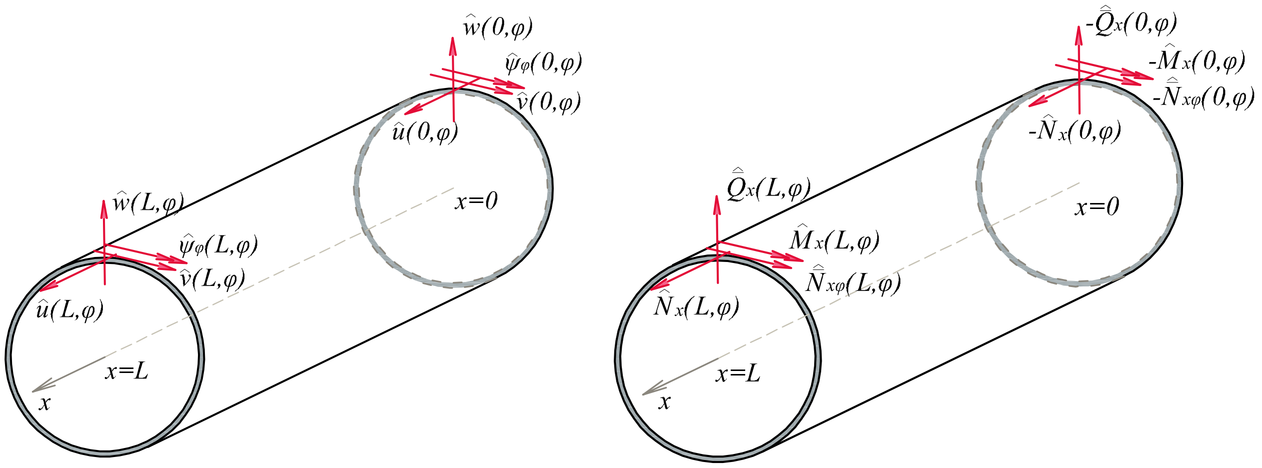
3 DINAMIČKA MATRICA KRUTOSTI $\mathbf{K}_{Dm}$

Vektor pomeranja $\hat{\mathbf{q}}$ i vektor sila $\hat{\mathbf{Q}}$ sadrže pomeranja i rotacije, odnosno sile i momente, na konturama ljuske $x=0$ i $x=L$:

3 DYNAMIC STIFFNESS MATRIX $\mathbf{K}_{Dm}$

The displacement vector $\hat{\mathbf{q}}$ and the force vector $\hat{\mathbf{Q}}$, contain the displacements and rotations, i.e. the forces and moments at the boundaries $x=0$ and $x=L$, respectively:

$$\begin{aligned}\hat{\mathbf{q}}^T&=\left[\hat{u}(0,\varphi)\ \hat{v}(0,\varphi)\ \hat{w}(0,\varphi)\ \hat{\psi}_\varphi(0,\varphi)\ \hat{u}(L,\varphi)\ \hat{v}(L,\varphi)\ \hat{w}(L,\varphi)\ \hat{\psi}_\varphi(L,\varphi)\right]\\ \hat{\mathbf{Q}}^T&=\left[-\hat{N}_x(0,\varphi)\ -\hat{\hat{N}}_{x\varphi}(0,\varphi)\ -\hat{\hat{Q}}_x(0,\varphi)\ -\hat{M}_x(0,\varphi)\ \hat{N}_x(L,\varphi)\ \hat{\hat{N}}_{x\varphi}(L,\varphi)\ \hat{\hat{Q}}_x(L,\varphi)\ \hat{M}_x(L,\varphi)\right]\end{aligned}\tag{18}$$



Slika 3. Komponente vektora pomeranja $\hat{\mathbf{q}}$ *i vektora sila* $\hat{\mathbf{Q}}$

Fig. 3. Components of the displacement and force vectors

Komponente vektora pomeranja $\hat{\mathbf{q}}$ i sila $\hat{\mathbf{Q}}$ prikazane su na slici (3) i definisane su sledećim izrazima:

The components of the vectors $\hat{\mathbf{q}}$ and $\hat{\mathbf{Q}}$, shown in Fig. (3) are defined by the following expressions:

$$\begin{aligned}
&\hat{u}(0,\varphi)=U_m(0)\cos(m\varphi) && \hat{u}(L,\varphi)=U_m(L)\cos(m\varphi)\\
&\hat{v}(0,\varphi)=V_m(0)\sin(m\varphi) && \hat{v}(L,\varphi)=V_m(L)\sin(m\varphi)\\
&\hat{w}(0,\varphi)=W_m(0)\cos(m\varphi) && \hat{w}(L,\varphi)=W_m(L)\cos(m\varphi)\\
&\hat{\psi}_\varphi(0,\varphi)=\Psi_{\varphi m}(0)\cos(m\varphi) && \hat{\psi}_\varphi(L,\varphi)=\Psi_{\varphi m}(L)\cos(m\varphi)\\
&\hat{\psi}_\varphi(x,\varphi)=\frac{\partial \hat{w}(x,\varphi)}{\partial x}
\end{aligned} \tag{19}$$

$$\begin{aligned}
&\hat{N}_x(x,\varphi)=\hat{N}_{xm}(x)\cos(m\varphi)=\left(\sum_{i=1}^{8}C_{i,m}N_{xi,m}e^{r_{im}x}\right)\cos(m\varphi)\\
&N_{xi,m}=\frac{D\nu\left(1+m\gamma_{i,m}\right)+aDr_{i,m}\delta_{i,m}-Kr_{i,m}^2}{a}\\
&\hat{\hat{N}}_{x\varphi}(x,\varphi)=\hat{N}_{x\varphi}+\frac{\hat{M}_{x\varphi}}{a}=\hat{\hat{N}}_{x\varphi m}(x)\sin(m\varphi)=\left(\sum_{i=1}^{8}C_{im}\bar{N}_{x\varphi i,m}e^{r_{im}x}\right)\sin(m\varphi)\\
&\bar{N}_{x\varphi i,m}=\frac{(1-\nu)\left[-aD\delta_{i,m}m+a^2D\gamma_{i,m}r_{i,m}+3Kr_{i,m}\left(\gamma_{i,m}+m\right)\right]}{2a^2}\\
&\hat{\hat{Q}}_x(x,\varphi)=\hat{Q}_x+\frac{1}{a}\frac{\partial \hat{M}_{x\varphi}}{\partial\varphi}=\hat{\hat{Q}}_{xm}(x)\cos(m\varphi)=\left(\sum_{i=1}^{8}C_{i,m}\bar{Q}_{xi,m}e^{r_{im}x}\right)\cos(m\varphi)\\
&\bar{Q}_{xi,m}=\frac{K\left[(1-\nu)\delta_{i,m}m^2+2a^2r_{i,m}^2\left(\delta_{i,m}-ar_{i,m}\right)\right]}{2a^3}+\frac{Kmr_{i,m}\left[(3-\nu)\gamma_{i,m}+2m(2-\nu)\right]}{2a^2}\\
&\hat{M}_x(x,\varphi)=\hat{M}_{xm}(x)\cos(m\varphi)=\left(\sum_{i=1}^{8}C_{i,m}M_{xi,m}e^{r_{im}x}\right)\cos(m\varphi)\\
&M_{xi,m}=\frac{K\left[m\nu\left(\gamma_{i,m}+m\right)+ar_{i,m}\delta_{i,m}-a^2r_{i,m}^2\right]}{a^2}
\end{aligned} \tag{20}$$

Vektori pomeranja $\hat{\mathbf{q}}_m$ i sila $\hat{\mathbf{Q}}_m$ koji sadrže amplitude pomeranja i rotacija, odnosno presečnih sila za $x=0$ i $x=L$ za *m*-ti harmonik su:

The new displacement and force vector, namely $\hat{\mathbf{q}}_m$ and $\hat{\mathbf{Q}}_m$, that contain the amplitudes of displacements and rotations, i.e. forces, on the boundaries $x=0$ and $x=L$ for m[th] harmonic are:

$$\left(\hat{\mathbf{q}}_m\right)^T = \left[U_m(0)\;\; V_m(0)\;\; W_m(0)\;\; \Psi_{\varphi m}(0)\;\; U_m(L)\;\; V_m(L)\;\; W_m(L)\;\; \Psi_{\varphi m}(L)\right]_{8x1}$$
$$\left(\hat{\mathbf{Q}}_m\right)^T = \left[-\hat{N}_{xm}(0)\;\; -\hat{\bar{N}}_{x\varphi m}(0)\;\; -\hat{\bar{Q}}_{xm}(0)\;\; -\hat{M}_{xm}(0)\;\; \hat{N}_{xm}(L)\;\; \hat{\bar{N}}_{x\varphi m}(L)\;\; \hat{\bar{Q}}_{xm}(L)\;\; \hat{M}_{xm}(L)\right]_{8x1} \quad (21)$$

Veza između vektora $\hat{\mathbf{q}}_m$ i vektora integracionih konstanti $\mathbf{C}_m$ uspostavlja se preko matrice $\mathbf{D}_m$, odnosno veza između vektora $\hat{\mathbf{Q}}_m$ i vektora integracionih konstanti se uspostavlja preko matrice $\mathbf{F}_m$, na sledeći način:

The vector $\hat{\mathbf{q}}_m$ is related to the vector of integration constants $\mathbf{C}_m$ by the matrix $\mathbf{D}_m$, while the vectors $\hat{\mathbf{Q}}_m$ and $\mathbf{C}_m$ are related through the matrix $\mathbf{F}_m$, as follows:

$$\hat{\mathbf{q}}_m = \mathbf{D}_m \mathbf{C}_m \quad (22)$$

$$\hat{\mathbf{Q}}_m = \mathbf{F}_m \mathbf{C}_m \quad (23)$$

gde su vektori integracionih konstanti $\mathbf{C}_m$ i matrice $\mathbf{D}_m$ i $\mathbf{F}_m$ jednake:

where the vector of integration constants $\mathbf{C}_m$ and matrices $\mathbf{D}_m$ and $\mathbf{F}_m$ are:

$$\mathbf{C}_m = \begin{bmatrix} C_{1,m} \\ C_{2,m} \\ C_{3,m} \\ C_{4,m} \\ C_{5,m} \\ C_{6,m} \\ C_{7,m} \\ C_{8,m} \end{bmatrix} \quad \mathbf{D}_m = \begin{bmatrix} \delta_{1,m} & \dots & \delta_{8,m} \\ \gamma_{1,m} & \dots & \gamma_{8,m} \\ 1 & \dots & 1 \\ -r_{1,m} & \dots & -r_{8,m} \\ \delta_{1,m}\cdot e^{r_{1,m}L} & \dots & \delta_{8,m}\cdot e^{r_{8,m}L} \\ \gamma_{1,m}\cdot e^{r_{1,m}L} & \dots & \gamma_{8,m}\cdot e^{r_{8,m}L} \\ e^{r_{1,m}L} & \dots & e^{r_{8,m}L} \\ -r_{1,m}\cdot e^{r_{1,m}L} & \dots & -r_{8,m}\cdot e^{r_{8,m}L} \end{bmatrix}_{8x8} \quad \mathbf{F}_m = \begin{bmatrix} -N_{x1,m} & \dots & -N_{x8,m} \\ -\bar{N}_{x\varphi1,m} & \dots & -\bar{N}_{x\varphi8,m} \\ -\bar{Q}_{x1,m} & \dots & -\bar{Q}_{x8,m} \\ -M_{x1,m} & \dots & -M_{x8,m} \\ N_{x1,m}\cdot e^{r_{1,m}L} & \dots & N_{x8,m}\cdot e^{r_{8,m}L} \\ \bar{N}_{x\varphi1,m}\cdot e^{r_{1,m}L} & \dots & \bar{N}_{x\varphi8,m}\cdot e^{r_{8,m}L} \\ \bar{Q}_{x1,m}\cdot e^{r_{1,m}L} & \dots & \bar{Q}_{x8,m}\cdot e^{r_{8,m}L} \\ M_{x1,m}\cdot e^{r_{1,m}L} & \dots & M_{x8,m}\cdot e^{r_{8,m}L} \end{bmatrix}_{8x8} \quad (24)$$

Ako se iz jednačine (32) izrazi $\mathbf{C}_m$ u funkciji od $\hat{\mathbf{q}}_m$ i zameni u jednačinu (33) dobija se veza između vektora $\hat{\mathbf{Q}}_m$ i $\hat{\mathbf{q}}_m$:

If vector $\mathbf{C}_m$ is expressed from the Eqs. (32) as a function of $\hat{\mathbf{q}}_m$ and replaced in the Eq. (33), the relation between vectors $\hat{\mathbf{Q}}_m$ and $\hat{\mathbf{q}}_m$ is obtained in the following form:

$$\hat{\mathbf{Q}}_m = \mathbf{K}_{Dm} \hat{\mathbf{q}}_m \quad (25)$$

gde je $\mathbf{K}_{Dm} = \mathbf{F}_m \left(\mathbf{D}_m\right)^{-1}$ dinamička matrica krutosti kružne cilindrične ljuske za *m*-ti harmonik. Red matrice $\mathbf{K}_{Dm}$ je 8.

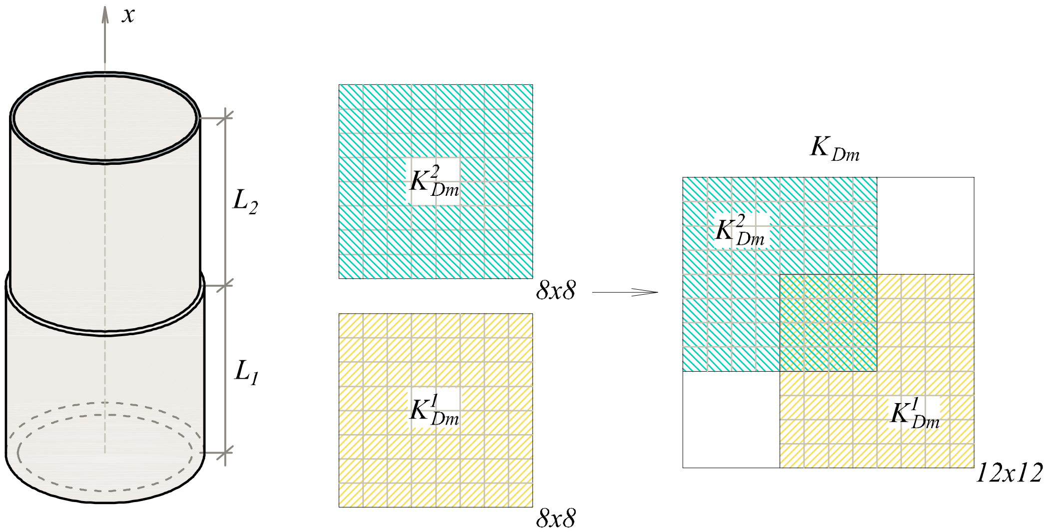
where $\mathbf{K}_{Dm} = \mathbf{F}_m \left(\mathbf{D}_m\right)^{-1}$ is the dynamic stiffness matrix of the circular cylindrical shells element for the m^{th} harmonic. The size of the matrix $\mathbf{K}_{Dm}$ is 8.

4 GRANIČNI USLOVI, SOPSTVENE FREKVENCIJE I OBLICI OSCILOVANJA

Kada je određena dinamička matrica krutosti kontinualnog elementa kružne cilindrične ljuske, globalna dinamička matrica krutosti sistema formira se na sličan način kao u MKE, samo što se elementi međusobno spajaju duž kontura, a ne u čvorovima. Na slici (4) prikazano je formiranje globalne matrice krutosti koja se sastoji od dva kontinualna elementa.

4 BOUNDARY CONDITIONS, NATURAL FREQUENCIES AND MODE SHAPES

When the dynamic stiffness matrix is determined, a global DS matrix of the structure can be obtained. The procedure is similar to that of FEM, except that continuous elements are connected at the boundaries, instead at nodes. Fig. (4) shows schematically the assembly procedure of two-element structure.



Slika 4. Formiranje globalne dinamičke matrice krutosti $\mathbf{K}_{Dm}$
Fig. 4. Assembly procedure of two-element structure

Kada je određena dinamička matrica krutosti sistema, granični uslovi se apliciraju brisanjem vrsta i kolona koje odgovaraju sprečenim pomeranjima. U numeričkim primerima biće prikazana rešenja samo za najčešće korišćene granične uslove:

- *slobodna kontura F*: sva tri komponentalna pomeranja i rotacija $\hat{\psi}_\varphi$ su različiti od nule,
- *shear diaphragm SD*: $v = w = 0$,
- *uklještena kontura C*: $u = v = w = \psi_\varphi = 0$.

Za slučaj kružno cilindrične ljuske uklještene na oba kraja (*C-C* granični uslovi) došlo bi do brisanja svih vrsta i kolona u matrici krutosti, pa za ovu kombinaciju graničnih uslova treba da se koriste najmanje dva kontinualna elementa.

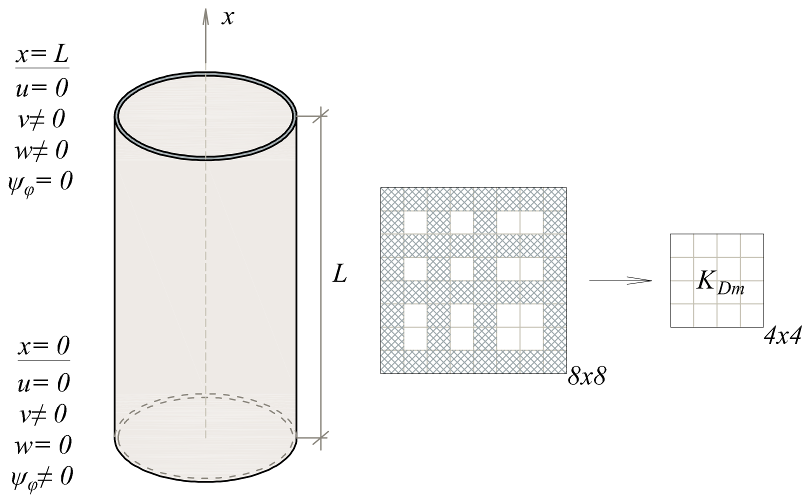
Na slici (5) šematski je prikazano apliciranje izabranih graničnih uslova.

When the global dynamic stiffness matrix is determined, the boundary conditions are applied by removing the rows and columns that correspond to the constrained displacements and rotations. In the numerical examples only the results for the most commonly used boundary conditions will be presented:

- *free F*: componential displacements and rotation $\hat{\psi}_\varphi$ are different from zero,
- *shear diaphragm SD*: $v = w = 0$,
- *clamped C*: $u = v = w = \psi_\varphi = 0$.

If a circular cylindrical shell with *C-C* boundary conditions is modelled with one continuous element, all rows and columns have to be deleted. Thus, for this combination of boundary conditions minimum two continuous elements have to be used.

The application of defined boundary conditions is shown in Fig. 5.



Slika 5. Apliciranje prikazanih graničnih uslova
Fig. 5. Application of the presented boundary conditions

Sopstvene frekvencije dobijaju se iz uslova da je $\det(\mathbf{K}_{Dm}) = 0$. Dinamička matrica krutosti je transcendentna, tako da se frekvencije za koje je $\det(\mathbf{K}_{Dm}) = 0$ dobijaju tehnikama pretraživanja. Za određivanje svojstvenih frekvencija, u radu su umesto nula determinante $\mathbf{K}_{Dm}$ traženi pikovi (peak) izraza $1/\log\det(\mathbf{K}_{Dm})$ [15]. Za tu svrhu je napisan kôd u programu *Matlab*.

Kada su određene sopstvene frekvencije, oblici oscilovanja dobijaju se na poznat način. Oblici oscilovanja ljuski nemaju prave čvorne linije, tj. linije na površi duž kojih su sva tri komponentalna pomeranja jednaka nuli. Umesto njih će se javiti čvorne linije duž kojih su dva pomeranja jednaka nuli, a treće pomeranje ima maksimalnu vrednost. U slučaju rotaciono-simetričnih vibracija (*m*=0) aksijalna, torziona i radijalna komponenta pomeranja potpuno su razdvojene, tako da se za svaku od njih dobijaju nezavisne čvorne linije.

5 NUMERIČKI PRIMERI

Verifikacija izvedenih tačnih dinamičkih matrica krutosti po Flügge-ovoj teoriji izvršena je upoređivanjem sopstvenih frekvencija dobijenih primenom Matlab programa s rešenjima dostupnim u literaturi, kao i s rezultatima programa Abaqus, na četiri primera. U svim primerima usvojena vrednost Poisson-ovog koeficijenta je ν = 0.3.

Primer 5.1

U ovom primeru je određeno prvih devet sopstvenih frekvencija u (*Hz*) za ljusku sa *C-C* graničnim uslovima i karakteristikama: $E = 210\ GPa$, $\rho = 78000\ kg/m^3$, $L = 20\ m$, $a = 1\ m$ i $h = 0.01\ m$. U Tabeli 1 data su rešenja dobijena primenom MDK po Flügge-ovoj teoriji, rezultati Zhang-a [16], kao i rezultati dobijeni pomoću programa Abaqus. Zhang je koristio Love–Timoshenko teoriju tankih ljuski, [17], i metod prostiranja talasa (*wave propagation approach*). Talasni broj u pravcu x-ose odredio je aproksimativno, kao talasni broj odgovarajuće grede sa sličnim graničnim uslovima. Za proračun u programu Abaqus korišćena su dva različita tipa konačnog elementa: *STRI3* (ravan konačni element sa šest čvorova) i *S4R* (konačni element ljuske sa četiri čvora koji uzima u obzir deformaciju smicanja). Znak “-“ u tabeli znači da je Zhang preskočio petu sopstvenu frekvenciju. Primenom samo jednog kontinualnog elementa dobijeno je slaganje s rezultatima MKE analize, gde je primenjeno 50526 *S4R*, odnosno 101052 *STRI3* konačnih elemenata.

The natural frequencies are obtained from the condition that $\det(\mathbf{K}_{Dm}) = 0$. Since the dynamic stiffness matrix is transcendent matrix, the natural frequencies are obtained using searching techniques. In this paper instead of finding zeros of the determinant $\mathbf{K}_{Dm}$, the peaks of the expression $1/\log\det(\mathbf{K}_{Dm})$ [15] are sought. Therefore, a computer program written in Matlab has been used.

When the exact natural frequencies are determined, the corresponding mode shapes are computed routinely. The mode shapes do not have true nodal lines, i.e. lines on the surface of a shell for which all displacement components are zero, but have lines along which two of displacement components are zero and the third has maximum values. In the case of axisymmetric vibration, the axial, circumferential and radial displacements are uncoupled, giving distinct nodal lines.

5 NUMERICAL RESULTS

The dynamic stiffness matrix based on the Flügge thin shell theory, formulated above, is verified through four numerical examples. The exact natural frequencies computed by the DSM are compared with the available analytical solution in the literature, as well as with the results from *Abaqus*. The adopted value of Poisson ratio ν is 0.3 in all examples.

Example 5.1

In this example, the first nine natural frequencies (*Hz*) of a circular cylindrical shell with *C-C* boundary conditions and the following properties: $E = 210\ GPa$, $\rho = 78000\ kg/m^3$, $L = 20\ m$, $a = 1\ m$ and $h = 0.01\ m$, are determined. Table 1 shows the results obtained by the *DSM* according to the Flügge thin shell theory, the results of Zhang [16], as well as the results obtained by Abaqus. Zhang used the Love-Timoshenko thin shell theory [17] and the *wave propagation* approach. The axial wave number is determined approximately as the wave number of an equivalent beam with similar boundary conditions. For the Abaqus models, two different finite elements are used: *STRI3* (flat element with six nodes) and *S4R* (4-node, quadrilateral shell element with reduced integration and a large-strain formulation). The sign "-" in the table means that Zhang missed the fifth natural frequency. By applying a single continuous element, total agreement with the results of FEM analysis has been obtained. The total number of elements in the FE model was 50526 for S4R elements, i.e. 101052 for STRI3 elements.

*Tabela 1. Prvih devet sopstvenih frekvencija u (*Hz*) za kružnu cilindričnu ljusku sa* C-C *graničnim uslovima:*
$E = 210\ GPa$ $\rho = 78000\ kg/m^3$, $L = 20\ m$, $a = 1\ m$, $h = 0.01\ m$, $\nu = 0.3$

Table 1. First nine natural frequencies (Hz) for a circular cylindrical shell with C-C boundary conditions:
$E = 210\ GPa$ $\rho = 78000\ kg/m^3$, $L = 20\ m$, $a = 1\ m$, $h = 0.01\ m$, $\nu = 0.3$

ton mode	[16]	*DSM*	*Abaqus*	
			S4R	*STRI3*
1	12.17	12.00	12.00	12.01
2	19.61	19.56	19.60	19.57
3	23.28	23.1	23.13	23.12
4	28.06	27.16	27.16	27.19
5	-	28.30	28.30	28.30
6	31.98	31.47	31.50	31.52
7	36.47	36.42	36.59	36.43
8	37.37	37.28	37.43	37.27
9	39.78	39.59	39.74	39.62

Primer 5.2

U Tabeli 2 prikazane su najniže bezdimenzionalne sopstvene frekvencije $\bar{\omega} = \omega a^2 \sqrt{\rho(1-\nu^2)/E}$ za kružnu cilindričnu ljusku sa *F-F* graničnim uslovima ($L/a = 20$, $h/a = 0.05$) za usvojeno *m*= 1, 2, 3, 4, 5, 6. Rezultati dobijeni primenom MDK po Flügge-ovoj teoriji, upoređeni su s rezultatima Zxang-a [18], kao i s rešenjima dobijenim pomoću programa Abaqus. Xiang je u svom radu za rešenje problema slobodnih vibracija koristio Goldenveizer–Novozhilov teoriju tankih ljuski ([19], [20]) i *state-space* metod za dobijanje homogenih diferencijalnih jednačina. U Abaqus-u je kružna cilindrična ljuska (*a = 1 m*) modelirana sa 101052 *STRI3* konačnih elemenata. Iz Tabele 2 vidi se dobro slaganje rezultata Xiang-a i MDK sa rezultatima dobijenim pomoću Abaqus-a. Relativna razlika u procentima između prikazanih rešenja govori da su razlike zanemarljive. Na slici (6) prikazani su oblici oscilovanja dobijeni primenom MDK i programa u Matlab-u, koji odgovaraju sopstvenim frekvencijama određenim u Tabeli 2.

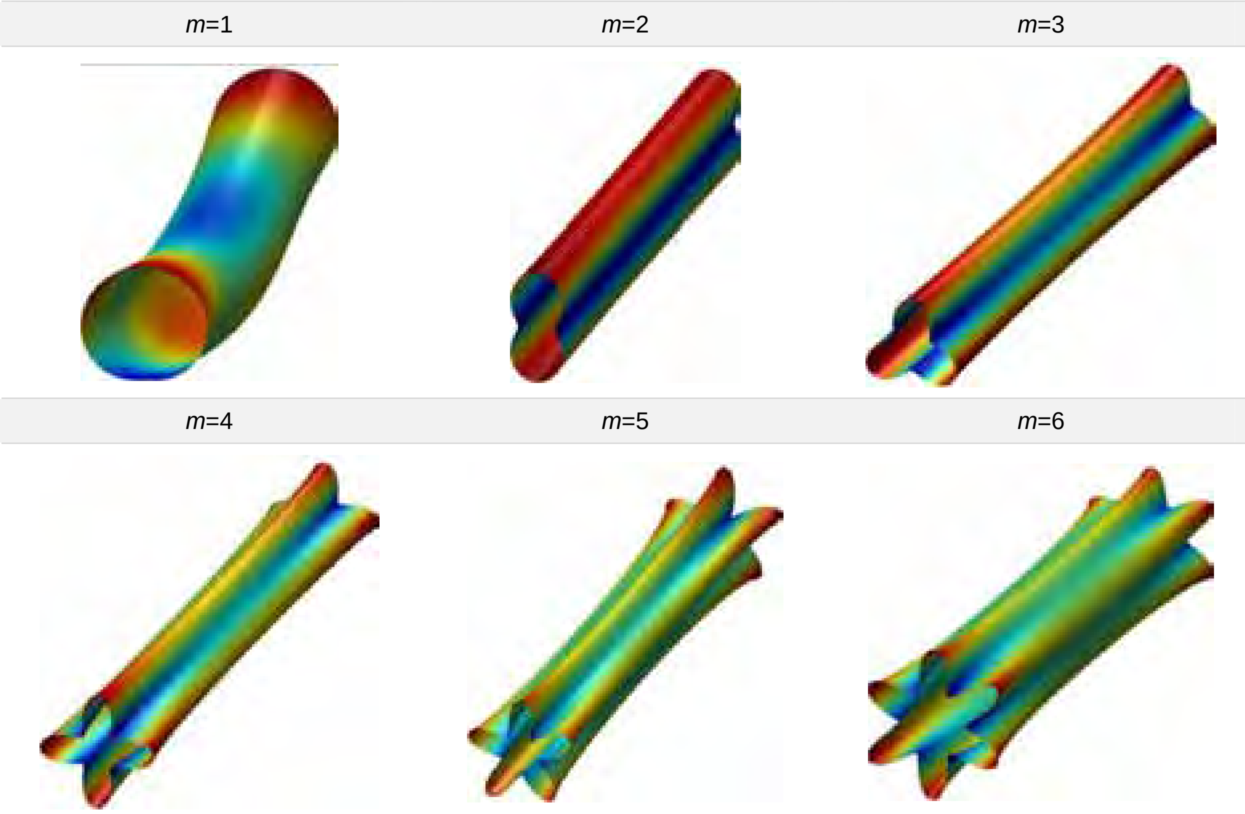
Example 5.2

Table 2 shows the first dimensionless natural frequencies $\bar{\omega} = \omega a^2 \sqrt{\rho(1-\nu^2)/E}$ for a circular cylindrical shell ($L/a = 20$, $h/a = 0.05$) with *F-F* boundary conditions and given *m*= 1,2,3,4,5,6. The results obtained by the DSM based on the Flügge thin shell theory are compared with the exact results of Zhang [18], as well as those obtained by Abaqus. Xhang used the Goldenveizer-Novozhilov thin shell theory ([19], [20]) and the state-space method to obtain the homogenous differential equations. In Abaqus, the circular cylindrical shell of radius *a = 1 m* has been modelled with 101052 STRI3 finite elements. Table 2 shows close agreement between the results obtained by Xhang, DSM and Abaqus. The relative differences between these results are negligible. In Fig. 6. the corresponding mode shapes obtained by the program written in the Matlab are presented.

Tabela 2. Najniža bezdimenzionalna sopstvena frekvencija $\bar{\omega} = \omega a^2 \sqrt{\rho(1-\nu^2)/E}$ *za usvojeno m kružne cilindrične ljuske sa FF graničnim uslovima:* $L/a = 20$, $h/a = 0.05$, $\nu = 0.3$

Table 2. Fundamental dimensionless natural frequency $\bar{\omega} = \omega a^2 \sqrt{\rho(1-\nu^2)/E}$ *of a circular cylindrical shell with F-F boundary conditions for a given m:* $L/a = 20$, $h/a = 0.05$, $\nu = 0.3$

m	[18]	Δ% [18] and *Abaqus*	*DSM*	Δ% *DSM* and *Abaqus*	*Abaqus* (*STRI3*)
1	0.035424	0.25	0.0355221	-0.03	0.03551
2	0.038683	0.03	0.0386885	0.01	0.03869
3	0.109366	-0.02	0.1093867	-0.04	0.10934
4	0.209644	-0.09	0.2096895	-0.12	0.20945
5	0.340330	-0.60	0.3390779	-0.23	0.33832
6	0.498824	-0.64	0.4973616	-0.35	0.49563



Slika 6. Oblici oscilovanja F-F kružne cilindrične ljuske koji odgovaraju najnižoj sopstvenoj vrednosti za usvojeno m

Figure 6. Fundamental modes shapes for a circular cylindrical shell ($L/a = 20$ *,* $h/a = 0.05$ *,* $\nu = 0.3$*) with F-F boundary conditions and given m*

Primer 5.3

U okviru ovog primera je demonstrirana primena metode dinamičke krutosti u analizi slobodnih vibracija kružne cilindrične ljuske sa skokovitom promenom debljine. Za usvojeno *m* određene su prve četiri bezdimenzionalne sopstvene frekvencije kružno cilindrične ljuske s jednostepenom promenom debljine i različitim graničnim uslovima. Rešenja dobijena primenom MDK po Flügge-ovoj teoriji upoređena su s rezultatima Zhang-a [21]. Zhang je koristio Flügge-ovu teoriju tankih ljuski, *state-space* metod i metod dekompozicije domena (*domain decomosition method*) da bi postavio uslove ravnoteže i kompatibilnosti na spoju dva segmenta. U Tabeli 3 prikazana su rešenja za ljuske sa sledećim graničnim uslovima i geometrijom: (1) *C-F* granični uslovi, $L/a = 1$ i $m = 1$ i 2, (2) *C-SD* granični uslovi, $L/a = 10$ i $m = 3$ i 4, (3) *C-C* granični uslovi, $L/a = 5$ i $m = 5$ i 6. Za sve ljuske je: $h_1/a = 0.01$, $h_2/h_1 = 0.5$ i $L_1/L = 0.5$, gde su h_1 i L_1 debljina i dužina prvog segmenta ljuske, h_2 je debljina drugog segmenta i L je ukupna dužina ljuske. U Tabeli 3 su, takođe, prikazane i relativne razlike u procentima između ova dve rešenja, koja ne prelaze 0.01 %.

Example 5.3

In this example, the application of the DSM in free vibration analysis of a stepped circular cylindrical shell has been demonstrated. The first four dimensionless natural frequencies of circular cylindrical shells with one-step thickness variation and with *C-F*, *C-SD* and *C-C* boundary conditions are calculated for a given *m*. Then results obtained by the DSM are compared with the results of Zhang [21]. Zhang used the Flügge thin shell theory, the state-space method and domain decomposition method in order to satisfy continuity requirements between shell segments. In Table 3, the results are presented for shells with following boundary conditions and geometry: (1) *C-F* boundary conditions, $L/a = 1$, $m = 1$,2, (2) *C-SD* boundary conditions, $L/a = 10$, $m = 3$,4 and (3) *C-C* boundary conditions, $L/a = 5$, $m = 5$,6. For all the shells applies the following: $h_1/a = 0.01$, $h_2/h_1 = 0.5$ and $L_1/L = 0.5$, where h_1 and L_1 are the thickness and the length of the first segment of the shell, h_2 is the thickness of the second segment and *L* is the total length of the shell. In Table 3 the relative differences in the percentage between the solutions are presented, which do not exceed 0.01%.

Tabela 3. Prve četiri bezdimenzionalne sopstvene frekvencije $\bar{\omega}=\omega a^2\sqrt{\rho\left(1-\nu^2\right)/E}$ *za usvojeno m kružne cilindrične ljuske s jednostepenom promenom debljine:* $h_1/a=0.01$, $h_2/h_1=0.5$, $L_1/L=0.5$, $\nu=0.3$

Table 3. First four dimensionless natural frequencies $\bar{\omega}=\omega a^2\sqrt{\rho\left(1-\nu^2\right)/E}$ *of a circular cylindrical shell with one-step thickness variation for a given m:* $h_1/a=0.01$, $h_2/h_1=0.5$, $L_1/L=0.5$, $\nu=0.3$

	mode	[21]	*DSM*	Δ% [20] - *DSM*	[21]	*DSM*	Δ% [21] - *DSM*
			$m=1$			$m=2$	
C-F $\frac{L}{a}=1$	1	0.637349	0.637356	0.00	0.409444	0.409447	0.00
	2	0.899615	0.899611	0.00	0.766426	0.766433	0.00
	3	0.948341	0.948335	0.00	0.911952	0.911948	0.00
	4	0.973919	0.973926	0.00	0.954440	0.954443	0.00
			$m=3$			$m=4$	
C-SD $\frac{L}{a}=10$	1	0.073550	0.073553	0.00	0.057247	0.057254	-0.01
	2	0.169777	0.169773	0.00	0.117799	0.117796	0.00
	3	0.275089	0.275093	0.00	0.194529	0.194532	0.00
	4	0.388211	0.388217	0.00	0.283104	0.283104	0.00
			$m=5$			$m=6$	
C-C $\frac{L}{a}=5$	1	0.038185	0.038187	-0.01	0.051992	0.051994	0.00
	2	0.057617	0.057618	0.00	0.064715	0.064712	0.01
	3	0.071696	0.071702	-0.01	0.089501	0.089506	-0.01
	4	0.085486	0.085492	-0.01	0.102022	0.102016	0.01

Primer 5.4

U poslednjem primeru određene su najniže bezdimenzionalne sopstvene frekvencije $\bar{\omega}=\omega a^2\sqrt{\rho\left(1-\nu^2\right)/E}$ za ljusku sa *SD-SD*, *C-C* i *F-F* graničnim uslovima koja ima dva, odnosno tri prstenasta međuoslonca na jednakim rastojanjima. Na mestu prstenastog međuoslonca sprečeno je radijalno pomeranje, tj. $w=0$. Posmatrani su slučajevi za koje je $L/a=$ 5 i 10 i $h/a=$0.05 i 0.005. Rezultati dobijeni primenom MDK po Flügge-ovoj teoriji upoređeni su s rezultatima Xiang-a [22], koji je koristio Goldenveizer-Novozhilov teoriju tankih ljuski, *state-space* metod i metod dekompozicije domena. Osnovne sopstvene frekvencije, zajedno sa odgovarajućim brojem *m*, date su u Tabeli 4. Relativna razlika u procentima, takođe prikazana u tabeli, ne prelazi 1.18%.

Example 5.4

In the last example, the fundamental dimensionless natural frequencies $\bar{\omega}=\omega a^2\sqrt{\rho\left(1-\nu^2\right)/E}$ for a shell with *SD-SD*, *C-C* and *F-F* boundary conditions and two, i.e. three equally spaced intermediate ring supports are computed. The radial displacement is prevented at the locations of ring supports, i.e. $w=0$. In the analysis $L/a=$ 5, 10 and $h/a=$0.05, 0.005 are adopted.

The results obtained by the DSM using the Flügge theory are compared with the results of Xiang [22], who used the Goldenveizer-Novozhilov thin shell theory, the state-space technique and the domain decomposition method. Fundamental frequencies, along with the corresponding circumferential number *m*, are given in Table 4. The relative differences in percentage, which is also presented in Table 4, do not exceed 1.18%.

Tabela 4. Najniže bezdimenzionalne sopstvene frekvencije $\overline{\omega} = \omega a^2 \sqrt{\rho\left(1-\nu^2\right)/E}$ *za kružnu cilindričnu ljusku sa dva, odnosno tri prstenasta međuoslonca (*$w = 0$*) na jednakim međusobnim rastojanjima,* $\nu = 0.3$

Table 4. Fundamental dimensionless natural frequencies $\overline{\omega} = \omega a^2 \sqrt{\rho\left(1-\nu^2\right)/E}$ *of a circular cylindrical shell with two and three equally spaced intermediate ring supports (*$w = 0$*):* $\nu = 0.3$

BC	$\frac{L}{a}$	$\frac{h}{a}$	dva međuoslonca / *2 supports*				tri međuoslonca / *3 supports*			
			[22]	*DSM*	Δ%	*m*	[22]	*DSM*	Δ%	*m*
SD-SD	5	0.005	0.0973340	0.09736141	-0.03	(7)	0.13052900	0.13056499	-0.03	(8)
		0.05	0.3066290	0.30744889	-0.27	(4)	0.39346800	0.39311378	0.09	(1)
	10	0.005	0.04757610	0.04758199	-0.01	(5)	0.06529310	0.06529980	-0.01	(6)
		0.05	0.14468164	0.14504722	-0.25	(3)	0.18596200	0.18648681	-0.28	(3)
C-C	5	0.005	0.10608300	0.10609919	-0.02	(7)	0.13655000	0.13658628	-0.03	(8)
		0.05	0.31320200	0.31385083	-0.21	(3)	0.40571000	0.40674818	-0.26	(4)
	10	0.005	0.05294030	0.05294578	-0.01	(5)	0.06864620	0.06865649	-0.01	(6)
		0.05	0.15388300	0.15425217	-0.24	(3)	0.19404800	0.19454980	-0.26	(3)
F-F	5	0.005	0.04372600	0.04367161	0.12	(5)	0.06023600	0.06017825	0.10	(5)
		0.05	0.11563700	0.11540796	0.20	(1)	0.17655500	0.17631291	0.14	(1)
	10	0.005	0.02093890	0.02091878	0.10	(3)	0.02938180	0.02934511	0.12	(4)
		0.05	0.0551552	0.05450301	1.18	(2)	0.11356600	0.11343547	0.11	(1)

6 ZAKLJUČAK

U ovom radu je formulisana dinamička matrica krutosti elementa kružne cilindrične ljuske. Dinamička matica krutosti određena je na osnovu tačnog rešenja sistema parcijalnih diferencijalnih jednačina kojima je definisan problem slobodnih vibracija po Flügge-ovoj teoriji. Izvedena dinamička matrica krutosti implementirana je u za tu svrhu napisani Matlab program za sračunavanje svojstvenih frekvencija i oblika oscilovanja kružne cilindrične ljuske i sistema kružnih cilindričnih ljuski. Primenom programa određene su svojstvene frekvencije za više karakterističnih primera. Dobijeni rezultati su upoređeni s dostupnim analitičkim rezultatima iz literature, kao i s rezultatima dobijenim primenom metoda konačnih elemenata i programa Abaqus. Analiza rezultata pokazala je odlično slaganje sa analitičkim rešenjima i s rešenjima dobijenim primenom programa Abaqus.

Prednost primene dinamičke matrice krutosti je očigledna: (1) za ljuske konstantnog preseka i s proizvoljnim graničnim uslovima na krajevima, dovoljno je koristiti samo jedan element, izuzev kod obostrano uklještene ljuske gde je potrebno koristiti dva kontinualna elementa; (2) za ljuske promenljive debljine i za ljuske s međuosloncima potreban je samo jedan

6 CONCLUSION

In this paper the dynamic stiffness matrix for a circular cylindrical shell element is formulated. Dynamic stiffness matrix is developed using the exact solution of the governing differential equations of free vibration according to the Flügge thin shell theory. The derived dynamic stiffness matrix is implemented in a Matlab program for calculation of natural frequencies and mode shapes of circular cylindrical shells and circular cylindrical shell assemblies. For several numerical examples natural frequencies are calculated using this program. The obtained results are validated against the available analytical results in the literature, as well as the results of the FE program Abaqus.

The analyses of the obtained results show excellent agreement with analytical solutions as well as the results obtained by Abaqus. The advantage of using the dynamic stiffness matrix is obvious: (1) for shells of constant cross section and arbitrary boundary conditions at the ends, it is sufficient to use only one DS element, except for shells clamped at both side, where it is necessary to apply two continuous elements; (2) for shells with stepped thickness variation and shells with intermediate ring supports it is necessary to apply one element for each segment of the shell. Contrary to the DSM in the element za svaki segment ljuske. Za razliku od MDK u MKE je potrebno mnogo elemenata (preko 100000) da bi se postigla željena tačnost.

Prikazani rezultati demonstrirali su glavnu prednost MDK u odnosu na MKE, a to je značajno smanjenje veličine modela i visoka preciznost rezultata u širokom frekventnom opsegu.

FEM a significant number of elements (over 100000) have to be applied in numerical model in order to achieve the desired accuracy. Presented results demonstrate the main advantages of DSM in comparison with FEM - a significant reduction of the size of model and highly accurate results in a wide frequency range.

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REZIME

METODA DINAMIČKE KRUTOSTI U ANALIZI VIBRACIJA KRUŽNE CILINDRIČNE LJUSKE

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U ovom radu korišćena je metoda dinamičke krutosti za analizu slobodnih vibracija kružne cilindrične ljuske. Dinamička matrica krutosti formulisana je na osnovu tačnog rešenja sistema diferencijalnih jednačina problema slobodnih vibracija po Flügge-ovoj teoriji ljuski. To je frekventno zavisna matrica koja u sebi, pored krutosti, sadrži uticaj inercije i prigušenja. Izvedena dinamička matica krutosti implementirana je u za tu svrhu napisani Matlab program za određivanje sopstvenih frekvencija i oblika oscilovanja kružne cilindrične ljuske. Urađen je niz primera. Rezultati dobijeni primenom dinamičke matrice krutosti upoređeni su s rezultatima dobijenim pomoću komercijalnog programa zasnovanog na metodi konačnih elemenata Abaqus, kao i sa dostupnim analitičkim rezultatima iz literature.

Ključne reči: slobodne vibracije, dinamička matica krutosti, Flügge-ova teorija ljuski

SUMMARY

DYNAMIC STIFFNESS METHOD IN THE VIBRATION ANALYSIS OF CIRCULAR CYLINDRICAL SHELL

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In this paper the dynamic stiffness method is used for free vibration analysis of a circular cylindrical shell. The dynamic stiffness matrix is formulated on the base of the exact solution for free vibration of a circular cylindrical shell according to the Flügge thin shell theory. The matrix is frequency dependent and, besides the stiffness, includes inertia and damping effects. The derived dynamic stiffness matrix is implemented in the code developed in a Matlab program for computing natural frequencies and mode shapes of a circular cylindrical shell. Several numerical examples are carried out. The obtained results are validated against the results obtained by using the commercial finite element program Abaqus as well as the available analytical solutions from the literature.

Key words: free vibration, circular cylindrical shell, Flügge thin shell theory, dynamic stiffness matrix,