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Application of the dynamic stiffness method in the vibration analysis of stiffened composite plates

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Abstract

Composite laminates are nowadays extensively applied in many engineering disciplines. Free vibration characteristics of such structures are not always easy to predict by using conventional finite element method (FEM). As an alternative, the dynamic stiffness method (DSM) can be applied to predict free vibration characteristics of composite plate assemblies, especially in mid and high frequency ranges. Key feature of the DSM is the dynamic stiffness element (DSE) and its dynamic stiffness matrix, derived from the exact solution of the governing equations of motion in the frequency domain. Consequently, the structural discretization is influenced only by the change in the geometrical and/or material properties of the structure. The number of unknowns is significantly decreased in comparison with the FEM, without losing the accuracy and reliability of the results.
In the paper, the DSE based on the higher order shear deformation theory (HSDT) is applied to study free vibration analysis of composite stiffened plates. The numerical analysis has been carried out through an illustrative example in order to check the accuracy of the proposed method. The influence of side-to-thickness ratio on the free vibration characteristics of stiffened plate has been studied numerically. The results are validated using the available analytical data as well as with the FEM solutions.

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Keywords: dynamic stiffness method; stiffened plate; composite laminate; HSDT.

1. Introduction

Composite laminates are nowadays extensively applied in many engineering disciplines as structural components of aircraft wings, ship hulls and FRP bridges, amongst others. In these structures, usually constructed in the form of

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plates joined at different angles, both in-plane and bending modes of vibration have been coupled. Free vibration characteristics of such structures are not always easy to predict by using conventional finite element method (FEM) based on the polynomial shape functions and weak form solution of the corresponding elastodynamic problem. As an alternative, the dynamic stiffness method (DSM) can be applied to accurately and efficiently predict the free vibration characteristics of composite plate assemblies, especially in the mid and high frequency ranges. The DSM is referred as the strong form-based method formulated in the frequency domain. Key feature of the DSM is the dynamic stiffness element and the corresponding dynamic stiffness matrix derived from the exact solution of the governing equations of motion. Consequently, the structural discretization is influenced only by the change in the geometrical and/or material properties of the structure. The number of unknowns is decreased in comparison with the FEM, decreasing the computational time/memory cost without losing the accuracy and reliability of the results.

Initially, application of the DSM was limited to the free vibration analysis of one-dimensional structures (beams and bars) [1-4] and Levy-type plates having two opposite edges simply supported [5-9], for which the governing equations of motion could be solved analytically. For plates having arbitrary boundary conditions, the issues of assignment of continuous boundary conditions and analytical solution of the governing equations, have occurred. The above issues can be overcome by using the projection method, presented in the works of Kevorkian & Pascal [10] and Casimir et al. [11]. Recently, in a series of contributions, a number of dynamic stiffness elements have been developed and applied in the free vibration analysis of plate assemblies undergoing both in-plane [12-15] and transverse vibration [16-22], accounting for different plate theories, material orthotropy, multi-layer properties and arbitrarily assigned boundary conditions.

In this paper, previous authors' work [12, 19, 21-22] has been extended and applied in the free vibration analysis of composite stiffened plates with arbitrary boundary conditions. First, the dynamic stiffness matrix of composite plate element undergoing in-plane vibration has been developed. Afterwards, the developed dynamic stiffness matrices for both in-plane and transverse vibration [21] based on the higher order shear deformation theory (HSDT) are rotated using the rotation matrix, before they have been assembled into the global dynamic stiffness matrix of the stiffened plate. The numerical analysis has been carried out through an illustrative example in order to check the accuracy of the proposed method. The influence of side-to-thickness ratio has been studied. The results have been validated against the available analytical data as well as against the FEM solutions.

2. Dynamic stiffness formulation of the dynamic stiffness element undergoing in-plane vibration

In the paper, we consider an assembly of rectangular cross-ply (0/90) laminated composite plates, each having the dimensions $2a\times2b$ and being composed of n orthotropic layers. The HSDT implies the following assumptions: (i) all layers are perfectly bonded together, (ii) the material of each layer is homogeneous, orthotropic and linearly elastic, (iii) small strains and small rotations are assumed, (iv) inextensibility of the transverse normal is imposed, (v) the displacement field is approximated using a cubic variation of the in-plane displacements through the thickness of the plate, leading to more realistic warping of the cross section and quadratic variation of transverse shear strains and transverse shear stresses through each layer of the laminate, [23]. The previous assumptions eliminate the application of the shear correction factors.

The formulation of the dynamic stiffness element is conducted separately for the in-plane and transverse vibration of the laminated composite plate, starting from two independent sets of Euler-Lagrange equations of motion. While the dynamic stiffness matrix of composite plate undergoing transverse vibration has been already formulated in authors' previous work [21], the procedure for the development of the dynamic stiffness matrix of rectangular laminated composite plate element undergoing in-plane vibration will be briefly presented, as follows.

In the first step, the harmonic representation of the in-plane displacement components u_0 and v_0 is introduced:

$$u_0\left(x,y,t\right)=\sum\hat{u}_0\left(x,y,\omega\right)e^{i\omega t},\quad v_0\left(x,y,t\right)=\sum\hat{v}_0\left(x,y,\omega\right)e^{i\omega t} \tag{1}$$

where $\hat{u}_0$ and $\hat{v}_0$ are the amplitudes of the in-plane displacement components u_0 and v_0, defined in the frequency domain, while ω is the considered angular frequency. Having in mind that Eq. (1) is valid for all angular frequencies in the considered frequency range, the argument ω will be omitted in further representations.

Governing differential equations of motion for the rectangular plate element undergoing in-plane vibration are:

$$a_1 \frac{\partial^2 \hat{u}_0}{\partial x^2} + \frac{\partial^2 \hat{u}_0}{\partial y^2} + (a_3+1)\frac{\partial^2 \hat{v}_0}{\partial x \partial y} + k\hat{u}_0 = 0, \quad a_2 \frac{\partial^2 \hat{v}_0}{\partial y^2} + \frac{\partial^2 \hat{v}_0}{\partial x^2} + (a_3+1)\frac{\partial^2 \hat{u}_0}{\partial x \partial y} + k\hat{v}_0 = 0 \tag{2}$$

In Eq. (2), $a_1 = A_{11}/A_{66}$, $a_2 = A_{22}/A_{66}$, $a_3 = A_{12}/A_{66}$ and $k = I_0\omega^2/A_{66}$, while A_{ij} and I_0 are the material parameters and mass moment of inertia of the laminate, respectively, which are given in [21]. The in-plane forces, given in terms of the amplitudes of the displacement components in the frequency domain, are:

$$\hat{N}_x(x,y) = A_{66}\left[a_1 \frac{\partial \hat{u}_0}{\partial x} + a_3 \frac{\partial \hat{v}_0}{\partial y}\right], \quad \hat{N}_y(x,y) = A_{66}\left[a_3 \frac{\partial \hat{u}_0}{\partial x} + a_2 \frac{\partial \hat{v}_0}{\partial y}\right], \quad \hat{N}_{xy}(x,y) = A_{66}\left[\frac{\partial \hat{u}_0}{\partial y} + \frac{\partial \hat{v}_0}{\partial x}\right] \tag{3}$$

According to Gorman's superposition method [12, 16, 19-22], the amplitudes of displacement components $\hat{u}_0$ and $\hat{v}_0$ are split into four symmetry contributions: symmetric-symmetric (SS), anti-symmetric - anti-symmetric (AA), symmetric - anti-symmetric (SA), anti-symmetric - symmetric (AS):

$$\hat{u}_0^{ij}(x,y) = \sum_{m=0,1}^{\infty} C_m \cdot f_m^u(x) \cdot g_m^u(y), \quad \hat{v}_0^{ij}(x,y) = \sum_{m=0,1}^{\infty} C_m \cdot f_m^v(x) \cdot g_m^v(y) \tag{4}$$

where i,j = S,A, $f_m^u(x)$, $f_m^v(x)$, $g_m^u(y)$ and $g_m^v(y)$ are trigonometric functions depending on the type of symmetry contribution [12], while C_m are the integration constants. Using the superposition method, it is now possible to analyze only a single quarter of the plate, which significantly reduces the size of the corresponding dynamic stiffness matrices. By using the method of separation of variables, the general solution for each symmetry contribution is presented in the Fourier series form truncated to a finite number of terms (M).

The in-plane displacement vector $\hat{\mathbf{q}}_{ij}$ and the corresponding force vector $\hat{\mathbf{Q}}_{ij}$ for each symmetry contribution, along the boundaries $x=a$ and $y=b$ of the quarter segment of the rectangular plate element, are defined as:

$$\begin{aligned} \hat{\mathbf{q}}^{ij}(x,y) &= \left[\hat{u}_0^{ij}(a,y) \quad \hat{v}_0^{ij}(a,y) \quad \hat{u}_0^{ij}(x,b) \quad \hat{v}_0^{ij}(x,b)\right]^T \\ \hat{\mathbf{Q}}^{ij}(x,y) &= \left[\hat{N}_x^{ij}(a,y) \quad \hat{N}_{xy}^{ij}(a,y) \quad \hat{N}_{xy}^{ij}(x,b) \quad \hat{N}_y^{ij}(x,b)\right]^T \end{aligned} \tag{5}$$

These vectors are functions of spatial variables x and y and consequently they cannot be related explicitly as in the case of the one–dimensional elements. Discretization of $\hat{\mathbf{q}}_{ij}$ and $\hat{\mathbf{Q}}_{ij}$ can be accomplished by introducing the projection method [10, 11], which is based on the projection of the above vectors onto a set of projection functions, as explained in [12]. Instead of using vectors $\hat{\mathbf{q}}_{ij}$ and $\hat{\mathbf{Q}}_{ij}$, new projection vectors $\tilde{\mathbf{q}}_{ij}$ and $\tilde{\mathbf{Q}}_{ij}$ are introduced, whose components are the Fourier coefficients in the series expansion (projections of the force and displacement vectors).

Now, it is possible to relate the vectors $\tilde{\mathbf{Q}}_{ij}$ and $\tilde{\mathbf{q}}_{ij}$ through the dynamic stiffness matrix $\tilde{\mathbf{K}}_{\mathbf{D}}^{ij}$, for each symmetry contribution (i,j). Finally, the dynamic stiffness matrix for a completely free plate element is derived from the dynamic stiffness matrices $\tilde{\mathbf{K}}_{\mathbf{D}}^{ij}$ of each symmetry contribution, by using the transformation matrix [12].

3. Rotation procedure for stiffened composite plates

Transverse and in-plane vibrations of a single plate represent two independent states. The corresponding dynamic stiffness matrix can be written as:

$$\tilde{\mathbf{K}}_{\mathbf{D}} = \begin{bmatrix} \tilde{\mathbf{K}}_{\mathbf{D}t} & 0 \\ 0 & \tilde{\mathbf{K}}_{\mathbf{D}i} \end{bmatrix} \tag{6}$$

In (6), $\tilde{\mathbf{K}}_{\mathbf{D}t}$ and $\tilde{\mathbf{K}}_{\mathbf{D}i}$ are the dynamic stiffness matrices of composite plate element for transverse (***t***) and in-plane vibration (***i***), respectively [12, 21]. For stiffened plates (where the plates are perpendicular to each other, as shown in

Fig. 1), transverse vibration of a single plate cause the in-plane vibration of the corresponding perpendicular plate, and vice versa. Therefore, it is necessary to establish the relation between the projection vectors of displacements/forces in the local, and the corresponding projection vectors in the global coordinate system (c. s.). This is accomplished by using the rotation matrix $\mathbf{T_R}$:

$$\tilde{\mathbf{q}} = \mathbf{T_R} \cdot \tilde{\mathbf{q}}^*, \quad \tilde{\mathbf{Q}} = \mathbf{T_R} \cdot \tilde{\mathbf{Q}}^* \tag{7}$$

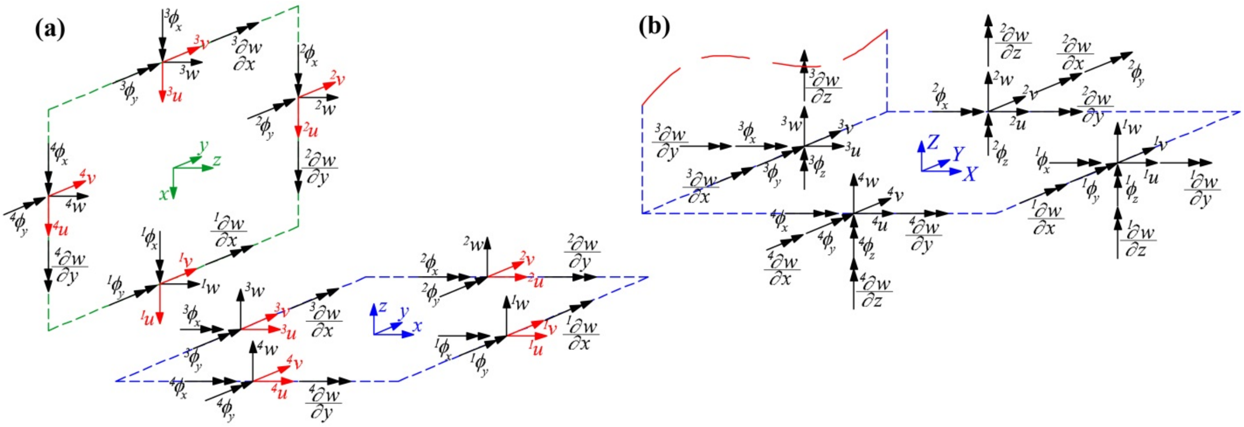

Fig. 1. (a)Single plates in the plate assembly with the DOFs in the local c. s.; (b) plate assembly with the DOFs in the global c. s.

According to the established relations between the projection vectors in the local and global c. s. (7), the dynamic stiffness matrix of composite plate in the global c. s. is derived as:

$$\tilde{\mathbf{K}}_{\mathbf{D}}^* = \mathbf{T}_{\mathbf{R}}^T \tilde{\mathbf{K}}_{\mathbf{D}} \mathbf{T_R} \tag{8}$$

Dynamic stiffness matrices of individual plates are assembled in the global dynamic stiffness matrix of plate assembly by applying the similar assembly procedure as in the conventional FEM [12, 21]. Note that the connection is established along plate boundaries, instead at nodes as in the FEM.

4. Numerical Example

To illustrate the applicability of the model, free vibration analysis has been performed for a square composite plate with an L stringer (see Fig. 2a). The thickness of each plate in the assembly is constant. Two different side-to-thickness ratios have been used (h/a = 0.010 and h/a = 0.005). All plates are made of orthotropic material having the following material properties: $E_1/E_2 = 40$, $G_{12}/E_2 = G_{13}/E_2 = 0.6$, $G_{23}/E_2 = 0.5$, $\nu_{12} = 0.25$. It is worth mentioning that the local x-axis of each plate (Fig. 2b) represents the global axis of each laminate in the assembly, with respect to the local material axes of each layer within a single plate. The 4-layers laminate is composed in a symmetric cross-ply stacking sequence (0/ 90/90/0).

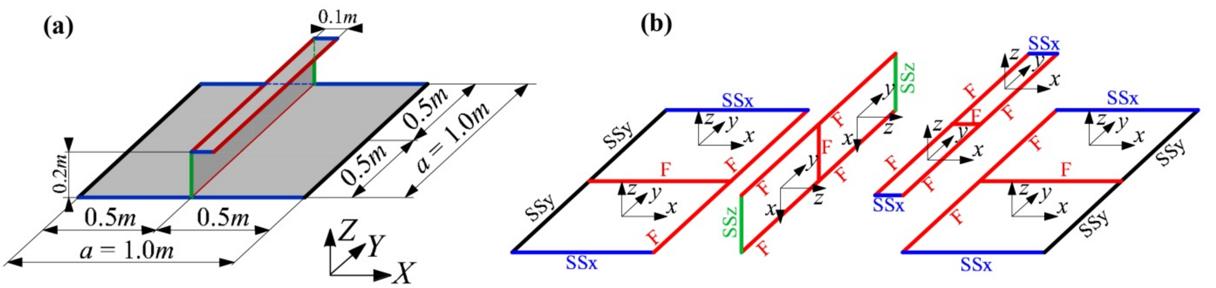

Fig. 2. (a) Stiffened composite plate with an L-stringer; (b) local coordinate systems and assigned boundary conditions.

The overall structure is simply supported along the boundaries (see Fig. 2b). The following boundary conditions have been assigned by constraining the following degrees of freedom in the global dynamic stiffness matrix: for the **SSx** case: $u=v=w=\phi_y=\phi_z=\partial w/\partial x=\partial w/\partial z=0$, for the **SSy** case: $u=v=w=\phi_x=\phi_z=\partial w/\partial y=\partial w/\partial z=0$, and finally for the **SSz** case: $u=v=w=\phi_x=\phi_y==\partial w/\partial x=\partial w/\partial y=0$.

The first 20 dimensionless natural frequencies have been computed using the proposed model and M=5 terms in the series expansion. The results are elaborated in Table 1 and compared with the results from the commercial software Abaqus (13000 S4R finite elements, element size = 0.01m), as well as with those obtained by DySAP [8]. In the proposed DSM formulation, the discretization has been made by using 8 dynamic stiffness elements, to avoid numerical instabilities which may arise for high plate aspect (a/b) ratios.

Table 1. First 20 dimensionless natural frequencies $\omega^* = \omega \cdot (a^2/h) \cdot (\rho/E_2)$ for a simply supported stiffened composite plate, considering different computational models and different side-to-thickness ratios

Mode	h/a = 0.005			h/a = 0.010	
	HSDT DSM, M = 5	Abaqus	Boscolo et al. [8]	HSDT DSM, M = 5	Abaqus
1	82.3097	82.2695	81.5	78.8540	78.7534
2	88.7186	88.6746	88.6	86.8965	86.5383
3	101.6619	101.6079	100.5	87.2106	86.7959
4	107.0655	107.0730	107.1	104.2380	104.0056
5	109.3274	109.3739	109.3	105.4947	105.3816
6	125.2867	125.2867	125.2	121.7053	121.5734
7	143.1310	143.0933	143.1	141.1832	140.9633
8	157.4566	157.4566	157.4	153.2469	151.3117
9	196.9150	197.1161	197.0	154.1894	153.9506
10	207.8478	207.9860	199.1	180.8301	178.4613
11	242.0283	240.2564	207.9	193.9619	193.5975
12	267.1610	267.6888	253.6	203.8265	203.4872
13	269.9256	269.5486	267.4	242.4681	238.5537
14	271.1823	271.1823	270.8	262.0088	261.4119
15	275.4548	275.9449	275.7	265.0876	265.2761
16	281.4867	281.5872	281.3	266.5956	265.2761
17	301.5929	301.1908	290.0	269.5486	268.9580
18	313.4053	311.8596	301.1	270.0513	269.4921
19	316.9239	316.2076	314.4	291.2256	290.4214
20	332.6318	332.7952	327.7	297.5717	296.0637

5. Conclusions

In this paper, the formulation of the dynamic stiffness matrix of a laminated composite stiffened plate assembly is presented as a superposition of the solution for the in-plane vibration and previously derived solution for the transverse vibration based on the HSDT [21]. The general solution of the governing equations is derived based on the superposition and projection methods. The proposed formulation provides highly accurate, reliable and robust computational model for the free vibration analysis of cross-ply composite stiffened plate assemblies, having arbitrary combinations of boundary conditions. The corresponding dynamic stiffness matrices (both for in-plane and transverse vibration) and a new module for the assembly procedure of the dynamic stiffness matrices, have been implemented into the original MATLAB code. The natural frequencies of stiffened composite plate are computed and validated against the existing analytical data and results from the commercial software Abaqus.

The results for the plate assembly with a/h = 0.005 clearly show that the proposed model is completely capable to predict the natural frequencies of stiffened composite plate. The results are in excellent agreement with the FEM results computed using Abaqus (the difference varies between 0.00% and 0.74%), while the average difference is

0.13%, which is practically negligible. When comparing the obtained results against the results calculated by Boscolo et al. [8] based on the FSDT, it is detected that some additional modes (i.e. ω_{10}* = 199.1) arise. When comparing the other 19 natural frequencies, the average difference is 0.94%, which is caused by the simplifications of the FSDT regarding the plate kinematics (i.e. using the shear correction factor, whose exact value is doubtful for laminar composites). Finally, the calculated natural frequencies are slightly higher in comparison with the FSDT-based model [8], which was expected. The additional comparison is made for *a/h* = 0.010. Excellent agreement with the results from Abaqus is achieved again (between 0.07% and 1.64%, average 0.42%). It is obvious that the accuracy of the Abaqus model decreases with increasing the plate thickness. This is already confirmed in [21-22].

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